FACTORIZATION OF SPANNING TREES ON FEYNMAN GRAPHS

(revised version)

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Abstract

In order to use the Gaussian representation for propagators in Feynman amplitudes, a representation which is useful to relate string theory and field theory, one has to prove first that each α-parameter (where α is the parameter associated to each propagator in the α-representation of the Feynman amplitudes) can be replaced by a constant instead of being integrated over and second, prove that this constant can be taken equal for all propagators of a given graph. The first proposition has been proven in one recent letter when the number of propagators is infinite. Here we prove the second one. In order to achieve this, we demonstrate that the sum over the weighted spanning trees of a Feynman graph G can be factorized for disjoint parts of G. The same can also be done for cuts on G, resulting in a rigorous derivation of the Gaussian representation for super-renormalizable scalar field theories. As a by-product spanning trees on Feynman graphs can be used to define a discretized functional space.

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1. Introduction

In the study of the relationship between field theories and string theories, the α-representation for Feynman graphs is a very useful tool [1-4]. In this representation one α-parameter is assigned to every propagator and the only integrations to be made are over these parameters, integration over the momenta circulating in the graph having already been made. This is therefore a very economical representation and it has quite a while ago been used to study the renormalization of field theories in the most accurate way [5]. However, it has another nice property ; writing a propagator of a scalar field theory as

$$[(P_i - P_j)^2]^{-1} = \int_0^\infty d\alpha \; exp[-\alpha(P_i - P_j)^2] \tag{1}$$

we see that the α-parameter is a sliding scale for Gaussians. If we fix α to some constant $\bar{\alpha}$ we have what we call a Gaussian representation (usually called a "Gaussian approximation") for the propagator. Now, discretized surface theories can be constructed using precisely the Gaussian representation in planar graphs amplitudes and taking $\bar{\alpha}$ to be the same quantity for *all* propagators of a graph. $\bar{\alpha}$ is then interpreted as proportional to the inverse of the slope of the Regge trajectories of the equivalent string theory [6]. In a recent letter [7] we proved that, indeed, once an integration over an overall scale was made, all the α_i's (α_i being the α-parameter of the propagator i) could be replaced by their mean-values $\bar{\alpha}_i$ which, in turn, were demonstrated to be O$(1/I)$, I being the total number of propagators of any one-particle, one vertex irreducible Feynman graph G, planar or non-planar, $I \to \infty$. This was done for any number of Euclidean dimensions where the theory was super-renormalizable. (When the theory is renormalizable we have to make the weak assumption that a logarithm coming from the renormalization of some sub-divergence is provoking only a shift in the coupling constant when the integration is made over the momenta of the legs of the sub-diverging part). However, in any case, there is a second step in the derivation which consists in proving that all the $\bar{\alpha}_i$'s can, in turn, be replaced by a single value $\bar{\alpha}$ for a given graph G. This could be demonstrated [7] provided

that the sum over spanning trees of G can be considered as a functional integral, i.e. that the *sum could be factorized* on disjoint domains of G. It is our purpose here to present a rigorous derivation of that statement and thereby ending the proof about the validity of the Gaussian representation (using an unique $\bar{\alpha}$).

In section 2 we present the basics of the $\bar{\alpha}$-representation for Euclidean scalar field theories. We give the general expression for the Feynman graph amplitude F_G of a graph G with I internal lines, L loops in terms of an integral over I α-parameters. This integral can be evaluated using the mean-value theorem which states that if a function f is continuous in its arguments $\{\alpha\}$, then

$$\int_{\omega} f = V \ f(\{\bar{\alpha}\}) \quad ,$$

where V is the volume of the connected domain ω over which the integration extends and $\{\bar{\alpha}\}$ a set of values of the α-parameters defining some point in ω. We expect the mean-values $\bar{\alpha}_i$ of the I α-parameters to be a priori different. Then, the central result of this article is to demonstrate that all $\bar{\alpha}_i$'s can be replaced by one *single* value $\bar{\alpha}$ without changing the value of Feynman amplitude (expressed as a function of the $\bar{\alpha}_i$'s via the mean-value theorem). This will amount to showing that the ratio of polynomials $Q_G(P_v, \{\bar{\alpha}_i\})$ defined in section 2 and appearing in the expression of F_G is indeed insensitive to that replacement. Isolating one particular $\bar{\alpha}_i$, Q_G can be set in the form

$$Q_G(P, \{\bar{\alpha}_i\}) = (\bar{e}_i/\bar{b}_i)(\bar{d}_i/\bar{e}_i + \bar{\alpha}_i)/(\bar{a}_i/\bar{b}_i + \bar{\alpha}_i)$$

where $\bar{a}_i/\bar{b}_i$ and $\bar{d}_i/\bar{e}_i$ are ratios of homogeneous polynomials in the mean-values of all α-parameters except α_i. The proof of the independence on the shift $\bar{\alpha}_i \to \bar{\alpha}$ will then translate into a proof of the equality of the ratios $\bar{a}_i/\bar{b}_i$ and $\bar{d}_i/\bar{e}_i$. Then, an important property of $\bar{a}_i$, $\bar{b}_i$, $\bar{d}_i$ and $\bar{e}_i$ is that they can be expressed as sums over products of $\bar{\alpha}_l$'s, l indexing propagators belonging to spanning trees of G (a spanning tree of G is a tree incident with all vertices of G). The ratio $\bar{a}_i/\bar{b}_i$ involves a sum over trees containing i, i.e.

$\bar{a}_i$, and a sum over trees not containing i, i.e. $\bar{b}_i$. The ratio $\bar{d}_i/\bar{e}_i$ is the ratio of a sum over trees containing i and cut at some other propagator over a sum over trees not containing i and cut at some other propagator (cutting means that the propagator is deleted from the tree, yielding a cut). Then, proving that $\bar{a}_i/\bar{b}_i = \bar{d}_i/\bar{e}_i$ amounts to proving that *the effect of cutting of a propagator can be factorized in the sum over trees.* This will be true if the structure of trees is such that their structure far from i is independent from their structure close to i. The next sections will be devoted to the proof that the sums over spanning trees can indeed be factorized over domains far apart on G.

In section 3 we restrict ourselves to the case of self-avoiding paths on G instead of trees. This is because, aside from simplicity in a first approach, there is always a self-avoiding path linking two vertices of G on any spanning tree of G. Thus, spanning trees can be built out of self-avoiding paths. We first give the general strategy for the proof of the equality $\bar{a}_i/\bar{b}_i = \bar{d}_i/\bar{e}_i$. Then, the following ratio

$$R_i(s_j) = \sum_P P(i, s_j) / \sum_P P(\bar{i}, s_j)$$

is proven to be independent of s_j if s_j is a vertex infinitely far from v_i on G, v_i being a vertex incident with the propagator i. (The notion of distance on G will be discussed later on. By infinitely far we mean that an infinite number of propagators separate v_i from s_j). $P(i, s_j)$ is a self-avoiding path linking v_i to s_j *going through* i. $P(\bar{i}, s_j)$ is also a self-avoiding path linking v_i to s_j *but not going through* i. The proof uses the evaluation of $R_i(s_j)$ as a mean-value in a volume V_j. In fact $R_i(s_{j+1})$ will turn out to be the average of $R_i(s_j)$. Letting $j \to \infty$, the averaging process, repeated an infinite number of times, removes the dependence on s_j of $R_i(s_j)$. This proof is essential as the same proof will be used to treat m-paths, i.e. paths with m connected parts, of which at least one of them is a path $P(i, s_j)$ or a path $P(\bar{i}, s_j)$. We also discuss some possible difficulties associated with the convergence of the averaging process.

In section 4, the main difficulty which could impede convergence is identified with the

fact that the ratio $R_i(s_{j+1})/R_i(s_j)_M$ can be infinitesimally close to one, $R_i(s_j)_M$ being an extremum value of $R_i(s_j)$ when the weight for $R_i(s_j)_M$ is infinite with respect to the sum of the weights for all other $R_i(s_j)$.

We solve this difficulty in the case of spanning trees in V_j directly. Then, s_j is replaced by $\{s_j\}$, a partition of the vertices of the border of V_j with $V_{j+1} - V_j$, where V_j is a volume which increases with j. In the averaging procedure $W_m^{m'}(\{s_j\}, \{s_{j+1}\})$ is the weight of $R_i^m(\{s_j\})$ in the evaluation of $R_i^{m'}(\{s_{j+1}\})$. The weight-ratio

$$\sum_{\{s_j\}_M} W_{m_M}^{m'}\left(\{s_j\}_M, \{s_{j+1}\}\right) / \sum_{\{s_j\}} W_m^{m'}\left(\{s_j\}, \{s_{j+1}\}\right)$$

is studied, where $\{s_j\}_M$ is a partition corresponding to $R_i^m(\{s_j\}_M)$, i.e. an extremum value of $R_i^m(\{s_j\})$ and $\{s_j\}$ any other partition. When the above weight-ratio becomes infinite we have a convergence problem in the case of spanning trees equivalent to the one for paths mentioned above. We then prove that if a constraint on the construction of the V_j's is imposed which, indeed, is easy to implement, the above weight-ratio takes the same value for all partitions $\{s_{j+1}\}$ of the vertices on the border of V_{j+1} with $G - V_{j+1}$.

This result allows to conclude that if the above weight-ratio is infinite, it is infinite whatever $\{s_{j+1}\}$ and then $R_i^{m'}(\{s_{j+1}\})$ is equal to $R_i^m(\{s_j\}_M)$, implying that convergence has been obtained. When, this weight-ratio is finite, then the convergence of the averaging process is not impeded and a unique value for $R_i^{m'}(\{s_{j+1}\})$ is obtained as $j \to \infty$.

In the last sub-section we show how the averaging process works in the case of spanning trees on G (instead of paths or multiple paths).

In section 5, the use of the above proof allow the proof of the factorization theorem. Section 6 will be the conclusion.

2. The α-representation

Here, we deal with scalar field theories in d Euclidean dimensions. We study one-line, one-vertex irreducible Feynman graphs with I internal lines (propagators), L loops, external momenta P_v and we take the coupling constant equal to - 1 in order to simplify. Then, F_G, the Feynman amplitude for a graph G and for a field of mass m reads [8]

$$F_G = h_0(4\pi)^{-dL/2} \int_0^{h_0} [\prod_{i=1}^{I} d\alpha_i]\ \delta(h_0 - \sum_i \alpha_i)\ [P_G(\alpha)]^{-d/2}.$$

$$\int_0^{\infty} d\lambda/\lambda\ \lambda^{I-dL/2}\ exp\{-\lambda[Q_G(P_v,\alpha) + m^2 h_0]\} \tag{2}$$

where $P_G(\alpha)$ is a homogeneous polynomial of degree L in the α_i's defined as

$$P_G(\alpha) = \sum_{\mathcal{T}} \prod_{l \notin \mathcal{T}} \alpha_l \tag{3}$$

where the sum runs over all the spanning tree $\mathcal{T}$ of G. (A spanning tree of G is a tree incident with every vertex of G). $Q_G(P,\alpha)$ is quadratic in the P_v's and is given by the ratio of a homogeneous polynomial of degree L+1 over $P_G(\alpha)$

$$Q_G(P_v,\alpha) = [P_G(\alpha)]^{-1} \sum_{\mathcal{C}} s_{\mathcal{C}} \prod_{l \in \mathcal{C}} \alpha_l \tag{4}$$

where the sum runs over all cuts $\mathcal{C}$ of L+1 lines that divide G in two connected parts $G_1(\mathcal{C})$ and $G_2(\mathcal{C})$, with

$$s_{\mathcal{C}} = (\sum_{v \in G_2(\mathcal{C})} P_v)^2 = (\sum_{v \in G_1(\mathcal{C})} P_v)^2 \tag{5}$$

(A cut $\mathcal{C}$ is obtained from a tree $\mathcal{T}$ by cutting off one line of $\mathcal{T}$. Then, the cut $\mathcal{C}$ will consist of all lines on G not on $\mathcal{T}$ *plus* the line of $\mathcal{T}$ which has been cut). We note that λ can be interpreted as an overall scale for the α-parameters ((2) indicates that $\sum_i \alpha_i = h_0$ where h_0 is arbitrary but taken equal to one in most circumstances). The integration over λ gives the overall divergence of F_G for a renormalizable theory. Here, we will limit ourselves to super-renormalizable theories, i.e. $I - dL/2$ will always be positive, giving a convergent integral

$$I_\lambda(Q_G) = \int_0^\infty d\lambda/\lambda \;\; \lambda^{I-dL/2} exp\{-\lambda[Q_G(P_v,\alpha)+m^2h_0]\}$$
$$= \Gamma(I-dL/2)\; [Q_G(P_v,\alpha)+m^2h_0]^{-(I-dL/2)} \tag{6}$$

Now, the spirit of the demonstration concerning the replacement of the α_i's by their mean-values $\bar{\alpha}_i$ consists in isolating the dependence of the integrand of F_G on one particular α_i and in using the mean-value theorem to perform the integration [7]. A discussion of the consistency of the result of this integration then shows that one should have, in any case [7],

$$\bar{\alpha}_i = O(h_0/I) \tag{7}$$

This property can easily be understood by considering the phase space for the I variables α_i, which can be found to be equal to $h_0^{I-1}/(I-1)! \sim (e\ h_0/I)^I$, leaving a phase space for each α_i of the order of $e\ h_0/I$.

Then, one has to show that indeed all $\bar{\alpha}_i$'s can be taken equal to some common value $\bar{\alpha}$. Consequently, we shall *define* $\bar{\alpha}$ by

$$\mathcal{N}_\mathcal{T}\bar{\alpha}^L = \sum_\mathcal{T}\prod_{l\notin\mathcal{T}}\bar{\alpha}_l = P_G(\bar{\alpha}) \tag{8}$$

where $\mathcal{N}_\mathcal{T}$ is the number of spanning trees on G. We see from (2) that the expression obtained for F_G by using the mean value theorem is

$$F_G = h_0(4\pi)^{-dL/2}[P_G(\bar{\alpha})]^{-d/2} I_\lambda[Q_G(P_v,\{\bar{\alpha}_i\})]\; h_0^{I-1}/(I-1)! \tag{9}$$

where the factor $h_0^{I-1}/(I-1)!$ is the volume of the phase space available for the α_i's. So, in fact, from (9) it is clear that our goal amounts to showing that $Q_G(P_v,\{\bar{\alpha}_i\})$ is not affected by the replacement $\bar{\alpha}_i \to \bar{\alpha}$. Let us write

$$P_G(\bar{\alpha}) = \bar{a}_i + \bar{b}_i\bar{\alpha}_i \tag{10}$$

$$\sum_{\mathcal{C}} s_{\mathcal{C}} \prod_{l\in\mathcal{C}} \bar{\alpha}_l = \bar{d}_i + \bar{e}_i\bar{\alpha}_i \tag{11}$$

where $\bar{a}_i, \bar{b}_i, \bar{d}_i$ and $\bar{e}_i$ do not contain any $\bar{\alpha}_i$ factor. As

$$Q_G(P_v, \{\bar{\alpha}_i\}) = (\bar{e}_i/\bar{b}_i)(\bar{d}_i/\bar{e}_i + \bar{\alpha}_i)/(\bar{a}_i/\bar{b}_i + \bar{\alpha}_i) \tag{12}$$

it is easily seen that if

$$\bar{d}_i/\bar{e}_i = \bar{a}_i/\bar{b}_i \quad , \tag{13}$$

the shift $\bar{\alpha}_i \to \bar{\alpha}$ will not affect $Q_G(P_v, \{\bar{\alpha}_i\})$. Repeating the reasoning for all $\bar{\alpha}_i$'s shows that $Q_G(P_v, \{\bar{\alpha}_i\})$ is invariant under the replacement of all the $\bar{\alpha}_i$'s by $\bar{\alpha}$ if (13) is true. It will be the purpose of the next sections to demonstrate that, up to vanishing corrections as $I \to \infty$, (13) is indeed true. Then, the shift of Log $[Q_G(P_v, \{\bar{\alpha}_i\})]$

$$\delta Q_G(P_v, \{\bar{\alpha}_i\})/Q_G(P_v, \{\bar{\alpha}_i\}) = Q_G^{-1} \sum_{i=1}^{I} \frac{\partial Q_G}{\partial \bar{\alpha}_i} \delta\alpha_i \tag{14}$$

will be vanishing, because $\delta\alpha_i \sim 1/I$ for any i ($\delta\alpha_i = \bar{\alpha} - \bar{\alpha}_i$). Then, defining $\Delta(\bar{\alpha})$ as

$$\Delta(\bar{\alpha}) = \prod_{l\in G} \bar{\alpha}_l \tag{15}$$

i.e. defined as the product of all $\bar{\alpha}_l$'s over G, we can write (see (3) and (10))

$$\bar{a}_i = \Delta(\bar{\alpha}) \sum_{\mathcal{T}\supset i} \prod_{l\in\mathcal{T}} \bar{\alpha}_l^{-1} \tag{16}$$

$$\bar{b}_i = \bar{\alpha}_i^{-1} \Delta(\bar{\alpha}) \sum_{\mathcal{T}\not\supset i} \prod_{l\in\mathcal{T}} \bar{\alpha}_l^{-1} \tag{17}$$

In an analogous way $\bar{d}_i$ and $\bar{e}_i$ can be written

$$\bar{d}_i = \Delta(\bar{\alpha}) \sum_{\mathcal{T}\supset i} \prod_{l\in\mathcal{T}} \bar{\alpha}_l^{-1} \sum_{\substack{k\in\mathcal{T}\\ k\neq i}} \bar{\alpha}_k \; s_{\mathcal{C}_k} \; \nu^{-1}(\mathcal{C}_k) \tag{18}$$

$$\bar{e}_i = \bar{\alpha}_i^{-1}\Delta(\bar{\alpha}) \sum_{\mathcal{T} \not\ni i} \prod_{l \in \mathcal{T}} \bar{\alpha}_l^{-1} \sum_{k \in \mathcal{T}} \bar{\alpha}_k \ s_{\mathcal{C}_k} \ \nu^{-1}(\mathcal{C}_k) \tag{19}$$

where $\nu(\mathcal{C}_k)$ counts the number of times the same cut $\mathcal{C}_k$ is obtained in cutting trees $\mathcal{T}$ at k, k being among the $\nu(\mathcal{C}_k)$ propagators binding the connected parts of G, $G_1(\mathcal{C}_k)$ and $G_2(\mathcal{C}_k)$ separated by $\mathcal{C}_k$.

To understand how (18) and (19) can be obtained from (11), let us recall that we are summing in (11) over all possible cuts $\mathcal{C}$ belonging to G. It is then, useful to note that the same cut $\mathcal{C}_k$, containing the propagator k, can be obtained by cutting different trees provided these trees have exactly the same structure in $G_1(\mathcal{C}_k)$ and $G_2(\mathcal{C}_k)$ and only in this case. That is, they will only differ by the propagator on them linking $G_1(\mathcal{C}_k)$ and $G_2(\mathcal{C}_k)$. Let us note by $\nu(\mathcal{C}_k)$ the number of propagators on G linking $G_1(\mathcal{C}_k)$ and $G_2(\mathcal{C}_k)$. Then, $\nu(\mathcal{C}_k)$ will count how many trees $\mathcal{T}$ can be cut to yield the same cut $\mathcal{C}_k$. Dividing by $\nu(\mathcal{C}_k)$ in (18) and (19) ensures that each cut is only counted once when cutting all possible trees yielding it. ($s_{\mathcal{C}_k}$ is defined by (5) where $\mathcal{C}$ is $\mathcal{C}_k$). Of course, all possible cuts are generated because k is taken to be any propagator of G on $\mathcal{T}$.

Comparing (18) and (19) to (16) and (17) respectively, we see that the sum $\sum_{k \in \mathcal{T}} \bar{\alpha}_k \ s_{\mathcal{C}_k} \ \nu^{-1}(\mathcal{C}_k)$ is the factor which distinguishes $(\bar{a}_i, \bar{b}_i)$ from $(\bar{d}_i, \bar{e}_i)$. If this factor can be factorized out of the sum over trees, the relation (13) will be obvious. However, this can only be done if the structure of the trees far from i is independent of their structure near i and when, in addition, k and i are far apart on G, i.e. if they are separated by an infinite number of propagators on G. When $I \to \infty$, most of the propagators of G will be far from i, so that we will be able to neglect in the sum over k, those k which are in a finite range of i. So our main goal will be, in fact, to show that *a factorization occurs in the sum of weighted trees for domains far apart on G.*

3. Construction of the spanning trees on G : paths on G

It will prove to be convenient for the following to rewrite (18) and (19) by inverting the summation order, $\mathcal{T}_k^i(\mathcal{C}_k)$ being a spanning tree which contains i and k and which cut at k gives a cut $\mathcal{C}_k$, containing k,

$$\bar{d}_i = \Delta(\bar{\alpha}) \sum_k \bar{\alpha}_k \sum_{\mathcal{T}_k^i(\mathcal{C}_k)} s_{\mathcal{C}_k} \ \nu^{-1}(\mathcal{C}_k) \prod_{l \in \mathcal{T}_k^i} \bar{\alpha}_l^{-1} \tag{20}$$

$$\bar{e}_i = \bar{\alpha}_i^{-1} \Delta(\bar{\alpha}) \sum_k \bar{\alpha}_k \sum_{\mathcal{T}_k(\mathcal{C}_k^i)} s_{\mathcal{C}_k^i} \ \nu^{-1}(\mathcal{C}_k^i) \prod_{l \in \mathcal{T}_k} \bar{\alpha}_l^{-1} \tag{21}$$

where $\mathcal{T}_k(\mathcal{C}_k^i)$ contains k but *not* i and gives a cut $\mathcal{C}_k^i$ when cut at k, $\mathcal{C}_k^i$ containing i and k. What we want to demonstrate now is that *for any k far apart from i on G*, defining

$$\bar{d}_{i,k} = \sum_{\mathcal{T}_k^i(\mathcal{C}_k)} s_{\mathcal{C}_k} \ \nu^{-1}(\mathcal{C}_k) \prod_{l \in \mathcal{T}_k^i} \bar{\alpha}_l^{-1} \tag{22}$$

$$\bar{e}_{i,k} = \bar{\alpha}_i^{-1} \sum_{\mathcal{T}_k(\mathcal{C}_k^i)} s_{\mathcal{C}_k^i} \ \nu^{-1}(\mathcal{C}_k^i) \prod_{l \in \mathcal{T}_k} \bar{\alpha}_l^{-1} \tag{23}$$

we have

$$\bar{d}_{i,k}/\bar{e}_{i,k} = \bar{a}_i/\bar{b}_i \tag{24}$$

up to terms which tend to zero as $I \to \infty$. The proof of (24) naturally entails the validity of (13) because $\bar{d}_i = \Delta(\bar{\alpha}) \sum_k \bar{\alpha}_k \ \bar{d}_{i,k}$ and $\bar{e}_i = \Delta(\bar{\alpha}) \sum_k \bar{\alpha}_k \ \bar{e}_{i,k}$ and also because those k within a "volume" V_j containing i and a number of propagators infinitesimal compared to the total number in G contribute to a negligible fraction of the sum, as will be shown at the end of section 5.

A - *General strategy for the proof of (24)*

We now give the general lines of the proof of the relation (24).

In the first place we consider G as embedded in a R^3-space embedded with a metric. Then, we consider a volume V_j in that space which contains the propagator i. When j is finite the number of propagators contained in V_j will be finite. As $j \to \infty$ the number of propagators contained in V_j will tend to infinity. We will however consider the number of propagators on G outside V_j infinite with respect to the number of those inside V_j, even as $j \to \infty$. The propagator k is taken to be outside V_j. Of course, in the sum over all k's some of them are inside V_j, but their number will be infinitesimal with respect to the total number of k's, i.e. the number of propagators in G. So the contribution of k's inside V_j will be negligible in the sum over them.

The reason we want to isolate V_j is that we shall see that inside V_j we can sum over sub-trees in it (when a spanning tree on G is cut by the border of V_j, the portion of that spanning tree inside V_j has no reason to be connected and is in general composed of several connected pieces that we call sub-trees) independently of the rest of the trees outside it provided the vertices on the border of V_j are partitioned in a definite way, each partition corresponding to a partition of sub-trees in V_j. So we get a *factorization* in the structure of trees on G, the sum over all trees in G being factorized into the sum over sub-trees in V_j times the sum over all sub-trees in $G - V_j$ *for a given partition of the border of* V_j *in sub-sets of vertices*, each sub-set being attached to the one sub-tree in V_j. Of course, sub-trees in $G - V_j$ (i. e. G minus all propagators and vertices in V_j) have to be compatible with those in V_j (not forming loops for instance) in that one should obtain trees in G altogether. However, again this implies only a restriction of the partition of vertices on the border of V_j, with respect to sub-trees in $G - V_j$ this time. So given the structure on the border of V_j, factorization of trees inside V_j and outside it holds.

However, we still have a dependence on the structure of the partition of the border of V_j. That is where comes a first and essential part of the proof.

Let us call $T(i, \{s_j\})$ those sets of sub-trees in V_j which contain the propagator i and

$T(\bar{i},\{s_j\})$ those sets of sub-trees in V_j which do not contain the propagator i, both being attached to a partition $\{s_j\}$ of the border of V_j. Let us define the ratio

$$R_i(\{s_j\}) = \sum_T T(i,\{s_j\}) / \sum_T T(\bar{i},\{s_j\}) \tag{25}$$

which is the ratio of the sum of the weights of sub-trees sets $T(i,\{s_j\})$ over the sum of the weights of sub-trees sets $T(\bar{i},\{s_j\})$. (Each propagator ℓ on a sub-tree brings a factor $\bar{\alpha}_\ell^{-1}$ in the weight of any sub-tree). Then, it will be proved in the next sub-section (for self-avoiding paths) and in the next section (for trees themselves) that as $j \to \infty$, i.e. when the radius of V_j grows to infinity, but with V_j still being infinitesimal with respect to G, $R_i(\{s_j\})$ tends to some value R_i^∞ independent of the partition $\{s_j\}$. Using the factorization property discussed above, this amounts to say that

$$\bar{a}_i/\bar{b}_i = R_i^\infty \bar{\alpha}_i \tag{26.a}$$

(see (16) and (17)). Now, the same argument can be used to derive also

$$\bar{d}_{i,k}/\bar{e}_{i,k} = R_i^\infty \bar{\alpha}_i \tag{26.b}$$

and thereby prove (24) if the additional structure of cutting throught k does not interfere with the inside of V_j, i.e. if $\nu(\mathcal{C}_k)$ and $s_{\mathcal{C}_k}$ are unaffected by the inside of V_j.

Let us denote by $\mathcal{S}_k$ the surface defined by the cut $\mathcal{C}_k$ going through k and dividing G into two separate pieces $G_1(\mathcal{C}_k)$ and $G_2(\mathcal{C}_k)$. ($\mathcal{S}_k$ cuts through all the propagators on G linking $G_1(\mathcal{C}_k)$ and $G_2(\mathcal{C}_k)$). If $\mathcal{S}_k$ does not go through V_j, the factorization property then trivially shows that the sums inside V_j are unaffected by $\mathcal{S}_k$, there is no interference. If $\mathcal{S}_k$ goes through V_j, $\nu(\mathcal{C}_k)$ counting the propagators in $\mathcal{S}_k$ will only be infinitesimally affected by the structure of sub-trees inside V_j, as well as $s_{\mathcal{C}_k}$, because the number of propagators in V_j is infinitesimal with respect with their total number in G. So there is

only an infinitesimal interference in this case and therefore the relations above prove the validity of (24).

In the next sub-sections, we simplify in a first approach, replacing trees on G by self-avoiding paths and prove in this case that the ratios $R_i(\{s_{j+1}\})$ are indeed averages of $R_i(\{s_j\})$. We also show how the factorization described above works. The averaging property means that

$$R_i(\{s_j\})_{min} \leq R_i(\{s_{j+1}\}) \leq R_i(\{s_j\})_{max} \tag{27}$$

where $R_i(\{s_j\})_{min}$ and $R_i(\{s_j\})_{max}$ are respectively the maximum and the minimum value of $R_i(\{s_j\})$. If, in the averaging, $R_i(\{s_j\})_{min}$ or $R_i(\{s_j\})_{max}$ only have a *finite weight relative to the sum of the weights of the other values of* $R_i(\{s_j\})$*, then it is clear that* $R_i(\{s_{j+1}\})$ *will be different from* $R_i(\{s_j\})_{min}$ *or* $R_i(\{s_j\})_{max}$, even having at least a *finite* (non-infinitesimal) difference with them. Then, we will have

$$R_i(\{s_{j+1}\})/R_i(\{s_j\})_{max} = 1 - \eta_1 \tag{28.a}$$

$$R_i(\{s_{j+1}\})/R_i(\{s_j\})_{min} = 1 + \eta_2 \tag{28.b}$$

η_1 and η_2 being positive and non-infinitesimal. As $j \to \infty$, the interval of variation of $R_i(\{s_j\})$ we tend to zero, and a value R_i^∞ independent of $\{s_j\}$ will be obtained for $R_i\{s_j\}$.

However, there may be a snag if the weight of $R_i(\{s_j\})_{max}$ or $R_i(\{s_j\})_{min}$ is infinite with respect to the sum of the weights of the other values of $R_i(\{s_j\})$ *for some* $\{s_{j+1}\}$, because (28) may not hold. In the next section, we study the weight-ratio

$$W_{m_2}^{m'}(\{s_j\}_2, \{s_{j+1}\})/W_{m_1}^{m'}(\{s_j\}_1, \{s_{j+1}\})$$

of weights corresponding to $R_i^{m_2}(\{s_j\}_2)$ and $R_i^{m_1}(\{s_j\}_1)$ in the evaluation of $R_i^{m'}(\{s_{j+1}\})$ and prove that when a certain constraint (easy to implement) on the construction of the

V_j's is imposed, it is independent of $\{s_{j+1}\}$. This allows us to conclude that either η_j is finitely different from zero or that when it is infinitesimal, it is so for any $\{s_{j+1}\}$, ensuring convergence.

B - *Self-avoiding paths*

Definition

A path $P(v_1, v_n)$ is defined as the succession of propagators (v_1v_2), (v_2v_3), $\cdots$, $(v_{n-1}v_n)$ linking v_1 to v_n, v_1, v_2, $\cdots$, v_n being n vertices on G. In a self-avoiding path v_1, v_2, $\cdots$, v_n are all different vertices.

A closed path is constructed when v_1 and v_n are the same vertex. A loop is a self-avoiding closed path. $\square$

The main tool we will be using now is the fact that, taking a vertex v_k at one end of the propagator k and a vertex v_i at one end of the propagator i, for each spanning tree $\mathcal{T}$ on G, there is one path, and only one, on $\mathcal{T}$, binding v_k and v_i. Furthermore, *this path is self-avoiding*, it goes through each vertex it is incident with only once. So the idea is to construct all spanning trees on G by beginning to construct all self-avoiding paths $P(v_i, v_k)$ on G binding v_i and v_k. Two spanning trees of G having a different path $P(v_i, v_k)$ are necessarily different. (If that were not the case, we would have two different paths $P(v_i, v_k)$ on the same tree, giving a loop on this tree which is forbidden). Of course, for every such path there exist many spanning trees obtained by sprouting branches out of the path. In fact, counting the number of trees associated with one path is the same as counting the number of ways branches can be sprouted out of this path. However, in a first step, we will concentrate our attention on the paths $P(v_i, v_k)$ themselves, taking into account the effect of branches later on, i.e. in the next section.

Now, let us consider the sum over all paths, each one being weighted by the product of all $\bar{\alpha}_l^{-1}$ belonging to the propagators along it. We denote by $P(i, s)$ a path $P(v_i, s)$ which

goes through the propagator i and by $P(\bar{i}, s)$ a path which *does not go through* i, both paths relating v_i and s. Then, the sums over all $P(i, v_k)$ and $P(\bar{i}, v_k)$ can be written (the sum over paths P, $\sum_P$, is a multiple sum, a summation being made for each path $P(s_p, s_{p+1})$),

$$\sum_P P(i, v_k) = \sum_P \sum_{s_i, \ldots, s_{lk-1}} P(i, s_1)P(s_1, s_2)...P(s_{lk-1}, v_k) \tag{29.a}$$

$$\sum_P P(\bar{i}, v_k) = \sum_P \sum_{s_i, \ldots, s_{lk-1}} P(\bar{i}, s_1)P(s_1, s_2)...P(s_{lk-1}, v_k) \tag{29.b}$$

$s_1, s_2, ..., s_{lk-1}$ belonging to the ensemble of the border-vertices of closed volumes $V_1, V_2, ..., V_{lk-1}$ such that

$$V_1 \subset V_2 \subset ... \subset V_{lk-1} \tag{30}$$

It is important to note that a path $P(s_j, s_{j+1})$ will *go out of* V_j *at* s_j *for the first time* but will otherwise be entirely contained in V_{j+1}. Moreover, $P(s_j, s_{j+1})$ can re-enter V_j and go out again at some vertex s'_j. We also will take V_1 to have a border at a finite distance of v_i, and the border of V_{j+1} to be at a finite distance of V_j *in terms of the minimum number of propagators separating them.*

Let us now define this notion of distance on a graph. We are going to embed G in an Euclidean space where each propagator has a definite length (this is the purpose of this embedding). Therefore, any path will have a definite length in this Euclidean space. The length of the path with the least number of propagators will be the distance between two vertices on G.

Let us also remark that, according to our above description, the distance between two vertices measured in Euclidean space and measured in the least number of propagators joining them have a priori *no monotonicity relation between them*, because the lengths of propagators can vary from one part to another part of G. We now define in a constructive way the volumes V_j.

C - *Construction of the V_j's*

We suppose that G is embedded in a 3-dimensional R_3 space. Then, we define the sphere S_1, i.e. the ensemble of points *within* a certain radius, centered at the vertex v_i, provided a metric has been defined in R_3. The radius of S_1 will be taken such that S_1 contains a finite number of propagators. In general, the two-dimensional border-surface of S_1 cut through propagators of G. We then deform continuously S_1 inwards in such a way that its border slides along the cut propagators until vertices attached to these propagators are met.

Definition

The border of V_1 is the deformed border of S_1. $\square$

As we suppose that v_i is inside S_1, it will also be inside V_1. Note that the border of V_1 may also contain entire propagators linking two vertices on its boundary. All the vertices and propagators belonging to the deformed sphere S_1 will belong to V_1.

Constructing V_2, we start with a sphere S_2 of radius larger than that of S_1. Then, we deform S_2 along the propagators cut by its border-surface until we meet the vertices attached to these propagators. Now, we are tempted to define V_2 in the same way as we did for V_1, by taking its border to coincide with the deformed sphere S_2. However, we will do so only in the case where the part of G between the borders of the deformed S_2 and S_1 is connected. In general, we expect that, in fact, *there will be several connected pieces of G between the borders of S_2 and S_1*. For each of these pieces we will have a corresponding connected piece of $V_2 - V_1$ (meaning V_2 from which V_1 has been subtracted) containing all the vertices and propagators of this connected piece of G. Let us consider in more details a given connected piece in $V_2 - V_1$.

Such a piece, let us call it $(V_2 - V_1)_C$ has a boundary formed by three pieces

i) a piece on the border of the deformed sphere S_2, which itself has a closed curve

C_2 as boundary,

ii) a piece on the border of the deformed sphere S_1, which itself has a closed curve C_1 as boundary,

iii) a cylindrical piece having C_1 and C_2 as boundaries. It should not be crossed by G.

In short, each connected $(V_2 - V_1)_C$ will enclose a corresponding connected part of G in $V_2 - V_1$, let us call it G_C. Then, C_2 encloses all the vertices and propagators of G_C on the surface of the deformed S_2. In the same manner C_1 encloses all the vertices and propagators of G_C on the border of V_1 (which according to the last definition is the surface of the deformed S_1). Note that there may be, in some cases, no vertex of G_C either in the surface enclosed by C_1 or in the surface enclosed by C_2, due to the topology of G (G may not go through the surface of the deformed S_2 at the border of $(V_2 - V_1)_C$, or through the piece of border of V_1 in common with that of $(V_2 - V_1)_C$).

Another condition we have on the boundary of $(V_2 - V_1)_C$ is that it should not cross the boundary of another connected piece in $V_2 - V_1$. In that way, two different connected volumes of $V_2 - V_1$ do not overlap.

This condition and the properties i), ii) and iii) above define the boun dary of one connected part $(V_2 - V_1)_C$.

For the following volumes $V_3, \cdots, V_j$, the constructive process we just described for V_2 repeats itself, each connected piece of $V_j - V_{j-1}$ being defined through its boundaries. It may happen that two successive V_{j-1}, V_j have some coinciding part of their borders (on a common part of the deformed S_{j-1} and S_j) because the density of propagators may be much larger in other parts of $V_j - V_{j-1}$. This ends our construction of the volumes V_j.

D - *An averaging theorem*

We then have an unequivocal definition for the paths $P(v_i, v_k)$. In ge neral the ratio

$P(i, s_1)/P(\bar{i}, s_1)$ depends on s_1. However, considering sums over paths, we want to prove that

$$R_i(s_j) = \sum_P P(i, s_j) / \sum_P P(\bar{i}, s_j) \tag{31}$$

tends to a value independent of s_j *as* $j \to \infty$. This is the clue for deriving (24) and thereby the factorization property. (It will also be proven in section 5 that when k is outside V_j, $j \to \infty$, for a given k, the factors $s_{\mathcal{C}_k}\nu^{-1}(\mathcal{C}_k)$ and $s_{\mathcal{C}_k^i}\nu^{-1}(\mathcal{C}_k^i)$ are equal). Now we want to prove the following lemma

Lemma 1

If $P(s_j, s_{j+1})$ never returns in V_j then,

$$R_i(s_j)_{min} \leq R_i(s_{j+1}) \leq R_i(s_j)_{max} \tag{32}$$

where $R_i(s_j)_{min}$ and $R_i(s_j)_{max}$ are respectively the minimum and the maximum values of $R_i(s_j)$ for those s_j coupled to s_{j+1} by at least one path $P(s_j, s_{j+1})$.

Proof

As a first remark, we can factorize the expression for $\sum\limits_P P(i, s_{j+1})$,

$$\sum_{s_j} \sum_P P(i, s_j) P(s_j, s_{j+1}) = \sum_{s_j} [\sum_P P(i, s_j)] \, [\sum_P P(s_j, s_{j+1})]$$

because the paths $P(s_j, s_{j+1})$ being entirely in $V_{j+1} - V_j$ never interact with the paths $P(i, s_j)$ contained in V_j. Therefore, using (31) we can write

$$\begin{aligned} R_i(s_{j+1}) = \{\sum_{s_j} [\sum_P P(i, s_j)] \, [\sum_P P(s_j, s_{j+1})]\} / \\ \{\sum_{s_j} \, [\, \sum_P P(\bar{i}, s_j)] \, [\, \sum_P P(s_j, s_{j+1})]\} \\ = \{\sum_{s_j} R_i(s_j) \, [\sum_P P(\bar{i}, s_j)] [\sum_P P(s_j, s_{j+1})]\} / \\ \{\sum_{s_j} \, [\sum_P P(\bar{i}, s_j)] [\sum_P P(s_j, s_{j+1})]\} \end{aligned} \tag{33}$$

We see that $R_i(s_{j+1})$ is an average of $R_i(s_j)$ for those s_j coupled to s_{j+1} and therefore (32) is true. $\Box$

Our goal is, of course, a repeated use of (32) and as j grows we expect $R_i(s_j)$ to become independent of s_j. However, we have restricted the paths $P(s_j, s_{j+1})$ to be in $V_{j+1} - V_j$ in order to avoid an interaction with the paths $P(v_i, s_j)$. If we allow such an interaction to occur, i.e. if $P(s_j, s_{j+1})$ returns in V_j, we have to distinguish between the different topologies of $P(s_j, s_{j+1})$. That is, we have to cut $P(s_j, s_{j+1})$ in parts which stay in V_j and parts which stay in $V_{j+1} - V_j$ *in order to be able to factorize the sum over paths.* Let us call s_j^0 the first vertex of $P(s_j, s_{j+1})$ and $s_j^1, s_j^3, ..., s_j^{2m-3}$ the vertices where $P(s_j, s_{j+1})$ re-enters V_j. (At s_j^2, s_j^4, ..., s_j^{2m-2} $P(s_j, s_{j+1})$ goes out of V_j). We denote by $\{s_j^l\}$ the ensemble $s_j^0, s_j^1, s_j^2, ..., s_j^{2m-2}$ and M_j the maximum number of connected parts of $P(i, s_{j+1})$ in V_j. Then, $\sum_P P(i, s_{j+1})$ can be written

$$\sum_P P(i, s_{j+1}) = \sum_{s_j^0} [\sum_P P(i, s_j^0)] \; [\sum_P P(s_j^0, s_{j+1})]$$
$$+ \sum_{m=2}^{M_j} \sum_{\{s_j^l\}} \sum_{\{s_j^{2l}, s_j^{2l-1}\}}$$
$$[\sum_P P(i, s_j^0) \prod_{l=1}^{m-1} P(s_j^{2l-1}, s_j^{2l})]$$
$$[\sum_P P(s_j^{2m-2}, s_{j+1}) \prod_{l=1}^{m-1} P(s_j^{2l-2}, s_j^{2l-1})] \tag{34}$$

where the paths $P(i, s_j^0)$, $P(s_j^{2l-1}, s_j^{2l})$ are contained in V_j and the paths $P(s_j^0, s_{j+1})$, $P(s_j^{2l-2}, s_j^{2l-1})$ and $P(s_j^{2m-2}, s_{j+1})$ have all their propagators contained in $V_{j+1} - V_j$. $\{s_j^{2\ell}, s_j^{2\ell-1}\}$ is the ensemble of couples $(s_j^{2\ell}, s_j^{2\ell-1})$ on $P(s_j, s_{j+1})$. The first term in (34) stands for the case where $P(s_j, s_{j+1})$ never returns in V_j and therefore corresponds to the case of Lemma 1 ($m = 1$). The following terms describe the case where $P(s_j, s_{j+1})$ returns in V_j ($m \geq 2$). Then, for a given m, $\{s_j^l\}$, $\{s_j^{2l}, s_j^{2l-1}\}$ the summation over paths has

been factorized into paths contained in V_j and paths contained in $V_{j+1} - V_j$. Notice that the couples (s_j^{2l}, s_j^{2l-1}), in fact, select an order among the s_j^l's, their order along the path $P(s_j^0, s_{j+1})$. However, the sum over all orders can be factorized as a sum over all orders of paths in V_j times a sum over all orders of paths in $V_{j+1} - V_j$. So, *we could dispense ourselves of tracking down the order of the s_j^l's through the couples (s_j^{2l}, s_j^{2l-1}). We do so only in order to make clear how the summation over paths is done.* Later on, *in the case of spanning trees this order will be meaningless* and it will not appear in the summation over all trees. Let us now define by (for $m \geq 2$)

$$R_i^m(s_j) = [\sum_P P(i, s_j^0) \prod_{l=1}^{m-1} P(s_j^{2l-1}, s_j^{2l})]/$$

$$[\sum_P P(\bar{i}, s_j^0) \prod_{l=1}^{m-1} \bar{P}(s_j^{2l-1}, s_j^{2l})] \tag{35}$$

the ratio of the part of $\sum_P P(i, s_{j+1})$ in V_j over the part of $\sum_P P(\bar{i}, s_{j+1})$ which also is in V_j ; m, $\{s_j^l\}$ and the couples $\{s_j^{2l}, s_j^{2l-1}\}$ are given.

$\bar{P}$ helps to distinguish the paths which are associated with $P(\bar{i}, s_j^0)$. Of course, $R_i(s_j) \equiv R_i^1(s_j)$.

We remark that the sums over P in (35) have *not* been factorized. This is, of course, because the paths in V_j have to avoid each other, so that a configuration of one of them affect the summation over the others. Then, we have the following theorem

Theorem 1

$$R_i^m(s_j)_{min} \leq R_i^{m'}(s_{j+1}) \leq R_i^m(s_j)_{max} \tag{36}$$

where $R_i^m(s_j)_{min}$ and $R_i^m(s_j)_{max}$ are respectively the minimum and the maximum values of $R_i^m(s_j)$ viewed as a function of m, $\{s_j^l\}$ and $\{s_j^{2l}, s_j^{2l-1}\}$, and for those s_j^l on at least one path $P(i, s_{j+1})$.

Proof

For the sake of simplicity of notation *we will make implicit the sum* $\sum_P$ *each time the symbol* P *appears.* Then define

$$P^m(i, s_j) = P(i, s_j^0) \prod_{l=1}^{m-1} P(s_j^{2l-1}, s_j^{2l}) \tag{37.a}$$

$$P^m(\bar{i}, s_j) = P(\bar{i}, s_j^0) \prod_{l=1}^{m-1} \bar{P}(s_j^{2l-1}, s_j^{2l}) \tag{37.b}$$

where the symbol $\bar{P}$ is for a path which is associated with $P(\bar{i}, s_j^0)$ and could differ from a path P because they are returning in V_1 where the paths $P(i, s_j^0)$ and $\bar{P}(i, s_j^0)$ are different. Writing $P^{m'}(i, s_{j+1})$ we obtain (we remark that, except for $P(i, s_j^0)$ and $\prod_{l=1}^{m-1} P(s_j^{2l-1}, s_j^{2l})$ which are confined in V_j, all other P's are in $V_{j+1} - V_j$) (see fig. 1)

$$\begin{aligned} P^{m'}(i, s_{j+1}) = & \sum_{s_j^0} P(i, s_j^0) P(s_j^0, s_{j+1}^0) \prod_{l=1}^{m'-1} P(s_{j+1}^{2l-1}, s_{j+1}^{2l}) \\ & + \sum_{m=2}^{M_j} \sum_{\{s_j^l\}} P(i, s_j^0) \prod_{l=1}^{m-1} P(s_j^{2l-1}, s_j^{2l}) \ . \\ & \sum_{\{l_t\}} \prod_{t=1}^{t_j} P(s_j^{2l_t-2}, s_j^{2l_t-1}) \ . \\ & \sum_{\{l_u\}} \prod_{u=1}^{u_j} P(s_j^{l_u}, s_{j+1}^{l_u}) \ . \\ & \sum_{\{l_v\}} \prod_{v=1}^{v_{j+1}} P(s_{j+1}^{2l_v-2}, s_{j+1}^{2l_v-1}) \end{aligned} \tag{38}$$

with

$$m - t_j + v_{j+1} = m' \tag{39.a}$$

$$\{s_j^l\} = \{s_j^{2l_t-2}\} \cup \{s_j^{2l_t-1}\} \cup \{s_j^{l_u}\} \tag{39.b}$$

The first term in (38) stands for the case where the connected component starting from v_i of $P^{m'}(i, s_{j+1})$, $P(i, s_j^0)$ $P(s_j^0, s_{j+1}^0)$, goes out of V_j at s_j^0 and stays in $V_{j+1} - V_j$, with all other connected components $P(s_{j+1}^{2l-1}, s_{j+1}^{2l})$ also staying in $V_{j+1} - V_j$. In this case, there is only one connected part in V_j, $P(i, s_j^0)$, of $P^{m'}(i, s_{j+1})$ and therefore this corresponds to $m = 1$. The following terms in (38) describe the other cases with m components ($m \geq 2$) in V_j : $P(i, s_j^0)$ with m - 1 other components $P(s_j^{2l-1}, s_j^{2l})$. The connected paths $P(s_j^{2l_t-2}, s_j^{2l_t-1})$ are never incident with the border of V_{j+1} and end up at the border of V_j. The connected paths $P(s_j^{lu}, s_{j+1}^{lu})$ start on the border of V_j and end up on the border of V_{j+1} while the connected paths $P(s_{j+1}^{2l_v-2}, s_{j+1}^{2l_v-1})$ start and end up on the border of V_{j+1} with no incidence with the border of V_j.

We remark that for $m' = 1$ we recover expression (34) because the products over the paths $P(s_{j+1}^{2l-1}, s_{j+1}^{2l})$ and $P(s_{j+1}^{2l_v-2}, s_{j+1}^{2l_v-1})$ do not exist in that case and the product over $P(s_j^{lu}, s_{j+1}^{lu})$ is replaced by one unique term $P(s_j^{2m-2}, s_{j+1})$ in (34).

The relation (39.a) is obtained by counting the number of vertices of $P^m(i, s_j)$ on the border of V_j which is equal to $2m$ - 1 and the number of vertices of $P^{m'}(i, s_{j+1})$ on the border of V_{j+1} which is equal to $2m'$ - 1 and calculating the difference $2(m'$ - $m)$. This difference comes from the connected paths $P(s_{j+1}^{2l_v-2}, s_{j+1}^{2l_v-1})$ which contributes $2v_{j+1}$ and $P(s_j^{2l_t-2}, s_j^{2l_t-1})$ which contributes $-2t_j$ to it. The same expression as (38) is obtained for $P^{m'}(\bar{i}, s_{j+1})$ by replacing $P(i, s_j^0)$ by $P(\bar{i}, s_j^0)$ and $P(s_j^{2l-1}, s_j^{2l})$ by $\bar{P}(s_j^{2l-1}, s_j^{2l})$. Then, writing (sums over P, again, are implicit)

$$R_i^{m'}(s_{j+1}) = P^{m'}(i, s_{j+1})/P^{m'}(\bar{i}, s_{j+1}) \tag{40}$$

and using (35) in it we obtain $R_i^{m'}(s_{j+1})$ as an average of $R_i^m(s_j)$ function of the s_j's and m, the average being the result of a sum over m, $\{s_j^l\}$ and the sets of couples $\{s_j^{2l-1}, s_j^{2l}\}$, $\{s_j^{2l_t-2}, s_j^{2l_t-1}\}$, $\{s_j^{lu}, s_{j+1}^{lu}\}$ and $\{s_{j+1}^{2l_v-2}, s_{j+1}^{2l_v-1}\}$ aside from the sum over paths P. However, once the set $\{s_j^l\}$ is given the partition of the paths in $V_{j+1} - V_j$ does not depend on that of the paths in V_j because there is no interaction between them. Therefore the sum over

all couples can be factorized out as it was already the case for the sum over the couples $\{s_j^{2l-1}, s_j^{2l}\}$. Consequently, the only variables which are necessary to retain are m and the set $\{s_j^l\}$ in the functional dependence of $R_i^m(s_j)$. As said earlier we retain the dependence on the couples $\{s_j^{2l}, s_j^{2l-1}\}$ in order to remind ourselves that we had to sum over all orders, this sum being, again, factorizable. Therefore the relation (36) follows. □

Again, a repeated use of (36) will allow us to make all the ratios $R_i^m(s_j)$ converge towards one value independent of s_j as $j \to \infty$.

However, we have to be careful about two problems potentially hampering the efficiency of this uniformization of $R_i^m(s_j)$.

a) As long as $R_i^m(s_j)$ has not converged, both inequalities (the same as (28.a) and (28.b))

$$R_i^{m'}(s_{j+1})/R_i^m(s_j)_{max} < 1 - \eta_1 \tag{41.a}$$

$$R_i^{m'}(s_{j+1})/R_i^m(s_j)_{min} > 1 + \eta_2 \tag{41.b}$$

should be satisfied, η_1 and η_2 being two positive non-infinitesimal constants.

b) G could have the topology of a tree-like structure or "polymer", i.e. many branches could stem out of V_1 and $V_{j+1} - V_j$ would be multiply connected. Remember that in Lemma 1 and Theorem 1, we have the restriction that s_j's should be on at least one path $P(i, s_{j+1})$. Then, a path going in some connected part of $V_{j+1} - V_j$ could never go in another connected part of $V_{j+1} - V_j$ because in order to do so, it would have to return in V_1 which may be impossible because it would have to go through vertices with which it is already incident. Then, due to this finite volume effect, most paths going along one branch of $V_{j+1} - V_j$ would never go in another branch. This leaves the possibility for $R_i^m(s_j)$ of evolving towards a different value along each branch of this tree-like structure, instead of a unique value as we wanted to show.

However, for spanning trees on G this difficulty is easily removed. The reason for

this is clear : a spanning tree of G is incident with every vertex of G. Then, we can think that a spanning tree on G is composed of one path $P(i, s_{j+1})$ and a myriad of paths stemming out of it, the ensemble of all paths going through all vertices of V_{j+1}, and then through all connected domains of $V_{j+1} - V_j$. For trees, a separate evolution of the quantity corresponding to $R_i^m(s_{j+1})$ along branches of $V_{j+1} - V_j$ is therefore prohibited. That is the essential difference between paths and spanning trees in our approach.

In the next section we also show how the problem a) is solved by insuring the validity of (41.a) and (41.b).

4. Trees on G

A - *Convergence of the iteration of mean-value operation*

Let us consider $R_i(s_{j+1})$ as given by the mean-value expression (33). For each s_j, $R_i(s_j)$ is multiplied by a weight $W(s_j, s_{j+1}) = \sum_P P(\bar{i}, s_j)\ P(s_j, s_{j+1}) / \sum_P P(\bar{i}, s_{j+1})$. Either (41.a) or (41.b) may be violated if and only if in the sum $\sum_{s_j} R_i(s_j) W(s_j, s_{j+1})$ the weight $W(s_j, s_{j+1})$ associated with $R_i(s_j)_{max}$ or $R_i(s_j)_{min}$ is infinite with respect to the sum of weights of any other value for $R_i(s_j)$. (By infinite, we mean that the ratios should be infinite).

Indeed, let us verify this and use (33), isolating the contribution of $R_i(s_j)_{max}$, s_{jm} being the vertex for which $R_i(s_{jm}) = R_i(s_j)_{max}$

$$R_i(s_{j+1}) = [R_i(s_j)_{max}\ W(s_{jm}, s_{j+1}) + \sum_{s_j \neq s_{jm}} R_i(s_j)\ W(s_j, s_{j+1})] /$$

$$[W(s_{jm}, s_{j+1}) + \sum_{s_j \neq s_{jm}} W(s_j, s_{j+1})] \quad . \tag{42}$$

We will take $R_i(s_j)/R_i(s_j)_{max}$ for $s_j \neq s_{jm}$ to have a *finite* (non-infinitesimal) difference with one (*otherwise convergence to a unique value is already achieved*) and simplify

the notation, defining

$$W_m = W\,(s_{jm}, s_{j+1}) \tag{43.a}$$

$$W = \sum_{s_j \neq s_{jm}} W(s_j, s_{j+1}) \quad . \tag{43.b}$$

Then,

$$R_i(s_{j+1})/R_i(s_j)_{max} = \left[W_m \; R_i(s_j)_{max} + W\bar{R}_i(s_j)\right] /$$

$$[(W_m + W)R_i(s_j)_{max}] \tag{44}$$

with

$$W\bar{R}_i(s_j) = \sum_{s_j \neq s_{jm}} R_i(s_j)\; W(s_j, s_{j+1}) \tag{45}$$

$\bar{R}_i(s_j)$ being the mean-value of $R_i(s_j)$ for $s_j \neq s_{jm}$. We have

$$\bar{R}_i(s_j) = (1 - \eta)\; R_i(s_j)_{max} \tag{46}$$

$\eta > 0$, and *non-infinitesimal*, following our assumption that $R_i(s_j)/R_i(s_j)_{max}$ is finitely different from one for $s_j \neq s_{jm}$. Hence, (44) gives

$$R_i(s_{j+1})/R_i(s_j)_{max} = 1 - \eta/(1 + W_m/W) \tag{47}$$

which shows that (41.a) *is satisfied provided the ratio* W_m/W *is not infinite.* Of course, the same sort of reasoning is also valid for showing that (41.b) is violated only if the weight of $R_i(s_j)_{min}$ is infinite with respect to the sum of the other weights.

Of course, it could also be that while having a finite weight, either $R_i(s_j)_{max}$ or $R_i(s_j)_{min}$ dominates the sum because the number of s_j's with $R_i(s_j) \neq$ (either $R_i(s_j)_{max}$ or $R_i(s_j)_{min}$) is infinitesimal with respect to the number of s_j's with $R_i(s_j) =$ (either $R_i(s_j)_{max}$ or $R_i(s_j)_{min}$). However, in this particular case, the range of variation of

$R_i(s_{j+1})$ would be infinitesimal and its convergence to value independent of s_{j+1} insured, which is what we want to demonstrate to be valid in any case.

Then, to obviate the difficulty mentioned above in the case of infinite weight-ratio W_m/W for either $R_i(s_j)_{max}$ or $R_i(s_j)_{min}$ we may show that taking s_{1j} and s_{2j} separated by a finite number of propagators in $V_{j+1} - V_j$ as well as in V_j the weight associated with $R_i(s_{2j})$ can be obtained from the weight associated with $R_i(s_{1j})$ by multiplying it by a finite factor. If $R_i(s_{1j})$ is $R_i(s_j)_{max}$ or $R_i(s_j)_{min}$ and $R_i(s_{2j})/R_i(s_{1j})$ is finitely different from one, the ratio W_m/W will then stay finite and the relations (41) will be valid. This can be easily understood because the corresponding paths $P(\bar{i}, s_{j+1})$ can be obtained from each other by a "local" deformation, i.e. by substituting only a finite number of propagators.

In the following, we will consider a proof of the validity of (41) inspired from this idea, but specific to trees. Therefore, we will need to translate the language adopted for paths and multi-paths into the one adopted for spanning trees and multiple-spanning trees. When we consider a spanning tree on G, the part of G contained in V_j will in general be a spanning m-tree in V_j, i.e. a spanning tree in V_j from which m propagators have been removed. The vertices of the border of V_j will be separated into m sub-sets $\{s_j^{m_c}\}$ (because each sub-tree has to be incident with the border of V_j on the border of the deformed S_j in order to be connected to the rest of the spanning tree on G), each sub-set $\{s_j^{m_c}\}$ belonging to one of the m sub-trees belonging to the spanning m-tree in V_j. There will also be sub-trees in $V_{j+1} - V_j$. Some (or all) of them will connect to a sub-tree in V_j to form, as a whole, a spanning m'-tree in V_{j+1}, i.e. a m'-tree which is incident with all the vertices contained in V_{j+1}. In the same way the vertices on the border of V_{j+1} will be divided into m' sub-sets, each belonging to a sub-tree of a spanning m'-tree in V_{j+1}. Therefore, in analogy with the multi-paths situation, we will write

$$R_i^m(\{s_j\}) = \sum_{T^m} T^m(i, \{s_j\}) / \sum_{T^m} T^m(\bar{i}, \{s_j\}) \tag{48}$$

where $T^m(i, \{s_j\})$ is the weight for a spanning m-tree in V_j going through the propagator

i and $T^m(\bar{i}, \{s_j\})$ is the weight of a spanning m-tree in V_j not going through i.

As for multi-paths the only information conveyed from the structure of m-trees in V_j to the sub-trees in $V_{j+1} - V_j$ is the partition $\{s_j\}$ of the vertices on the part of the border of V_j which is on the border of the deformed sphere S_j. This partition is common to the m-trees of weight $T^m(i, \{s_j\})$ and weight $T^m(\bar{i}, \{s_j\})$ and is the ensemble of all sub-sets $\{s_j^{m_c}\}$. We write in analogy with (33)

$$R_i^{m'}(\{s_{j+1}\}) = \sum_{\{s_j\}} R_i^m(\{s_j\}) \; W_m^{m'}(\{s_j\}, \{s_{j+1}\}) \tag{49.a}$$

$$W_m^{m'}(\{s_j\}, \{s_{j+1}\}) = \sum_{T^m} T^m(\bar{i}, \{s_j\}) \sum_{T^n} T^n(\{s_j'\}, \{s_{j+1}\})/N_{m'} \tag{49.b}$$

$$N_{m'} = \sum_{T^{m'}} T^{m'}(\bar{i}, \{s_{j+1}\}) \tag{49.c}$$

where $T^n(\{s_j'\}, \{s_{j+1}\})$ is the weight of a spanning n-tree in $V_{j+1} - V_j$ *which together with a m-tree in V_j forms a spanning m'-tree in V_{j+1}.* $\{s_j'\}$ may not be identical to $\{s_j\}$ because not all vertices in $\{s_j\}$ may be incident with a n-tree in $V_{j+1} - V_j$. $N_{m'}$ is the sum over the weights of all spanning m'-trees in V_{j+1}, $T^{m'}(\bar{i}, \{s_{j+1}\})$, which do not go through i and correspond to $\{s_{j+1}\}$.

Some partitions $\{s_j\}_M$ correspond to an extremum value $R_i^m(\{s_j\}_M)$ and to ensure the validity of (41) we have to ensure that the weight $W_{m_M}^{m'}(\{s_j\}_M, \{s_{j+1}\})$ corresponding to these partitions does not become infinite relative to the sum of weights of the other partitions. Or, alternatively if that case arises, showing that the ratio

$$\sum_{\{s_j\}_M} W_{m_M}^{m'}(\{s_j\}_M, \{s_{j+1}\} / \sum_{\{s_j\}} W_m^{m'}(\{s_j\}, \{s_{j+1}\}) \tag{50}$$

stays infinite whatever $\{s_{j+1}\}$ is also solves our problem. Then, we would already have convergence to a unique value $R_i^{m'}$ equal to $R_i^m(\{s_j\}_M)$. Our strategy will consist in

building a constructive procedure which allows us to calculate the ratio of two weights

$$W^{m'}_{m_2}(\{s_j\}_2, \{s_{j+1}\})/W^{m'}_{m_1}(\{s_j\}_1, \{s_{j+1}\}) \tag{51}$$

for two different partitions $\{s_j\}_1$ and $\{s_j\}_2$, the partition $\{s_{j+1}\}$ remaining the same. $\{s_{j+1}\}$ is the partition of vertices on the part of the border of V_{j+1} on the deformed sphere S_{j+1}. It consists in sub-sets $\{s^{m'_c}_{j+1}\}$, $(m'_c = 1, \cdots, m')$ of vertices on a sub-tree of a spanning m'-tree in V_{j+1}. Now, $\{s_j\}_2$ can always be obtained from $\{s_j\}_1$ by *a series of minimal modifications*, each of which consists in cutting in $V_{j+1} - V_j$ a self-avoiding path connecting two vertices s_{1j} and s_{2j} belonging to those forming $\{s_j\}$ instead of cutting a path staying V_j relating the same vertices s_{1j} and s_{2j}. Thus, in a minimal modification a propagator in $V_{j+1} - V_j$ on a m'-tree in V_{j+1} is replaced by a propagator in V_j, obtaining another m'-tree in V_{j+1} but with the same $\{s_{j+1}\}$ because s_{1j} and s_{2j} (and the vertices s_{j+1} connected to them) stay connected. The reverse operation is also considered as a minimal modification. We will discuss further below this minimal modification.

B - *Proof of the independence of the weight-ratio (51) on* $\{s_{j+1}\}$

We know that for spanning trees in V_{j+1}, given two vertices s_{1j} and s_{2j} of $\{s_j\}$, there are two possibilities :

i) they are not connected in V_j,

ii) they are connected in V_j by a self-avoiding path on the spanning tree.

Let us call such a path in V_j $P_1(s_{1j}, s_{2j})$. From such a path, spanning trees in V_{j+1} can be constructed by rooting branches on it, and then second branches rooted on the first branches, and then again branches rooted on these second branches, and so on until every vertex in V_{j+1} is incident with a branch. Let us suppose for a while that a self-avoiding path $P_2(s_{2j}, s_{1j})$ is entirely in $V_{j+1} - V_j$. From this path again spanning trees in V_{j+1} can be constructed in the same way as for $P_1(s_{1j}, s_{2j})$. However, on these last spanning trees s_{1j} and s_{2j} are disconnected in V_j. We remark that the succession of paths $P_1(s_{1j}, s_{2j})$

$P_2(s_{2j}, s_{1j})$ forms a loop $\mathcal{L}$ which crosses the border of V_j with $V_{j+1} - V_j$ at s_{1j} and s_{2j}. Then,

i) if we cut the loop $\mathcal{L}$ by removing one propagator $(v_1 v_2)$ on $P_1(s_{1j}, s_{2j})$, v_1 and v_2 being the end-vertices of the propagator, we get a self-avoiding path on $\mathcal{L}$, $P(v_2, v_1)$ from which spanning trees in V_{j+1} can be constructed in which s_{1j} and s_{2j} are not connected in V_j. $P(v_2, v_1)$ is considered as $P_2(s_{2j}, s_{1j})$ with two branches on $P_1(s_{1j}, s_{2j})$ rooted at s_{1j} and s_{2j}.

ii) If we cut the loop $\mathcal{L}$ by removing one propagator $(v'_1 v'_2)$ on $P_2(s_{2j}, s_{1j})$, v'_1 and v'_2 being the end-vertices of this propagator, we get a self-avoiding path $P(v'_1, v'_2)$ from which spanning trees in V_{j+1} can be constructed in which s_{1j} and s_{2j} *are connected in* V_j. $P(v'_1, v'_2)$ is considered as $P_1(s_{1j}, s_{2j})$ with two branches on $P_2(s_{2j}, s_{1j})$ rooted at s_{1j} and s_{2j}.

Cutting $(v_1 v_2)$ instead of $(v'_1 v'_2)$ on $\mathcal{L}$ *then defines a minimal modification* which modifies $\{s_j\}$. Calculating the weight of the sum of spanning m'-trees in V_{j+1} obtained by cutting $m' - 1$ propagators of the spanning trees constructed in i) we get a weight $W^{m'}_{m_1}(\{s_j\}_1, \{s_{j+1}\})$ if the cutting is made such as to preserve $\{s_j\}_1$ and $\{s_{j+1}\}$. Calculating the weight of the sum of spanning m'-trees in V_{j+1} obtained by cutting $m' - 1$ propagators of the spanning trees constructed in ii) we get a weight $W^{m'}_{m_2}(\{s_j\}_2, \{s_{j+1}\})$ if again the cutting is made such as to preserve $\{s_j\}_2$ and $\{s_{j+1}\}$. We will see that for a given $\mathcal{L}$ the ratio the respective contributions from i) and from ii) to the weights is easily obtained. (Here we have assumed that $\{s_j\}_1$ and $\{s_j\}_2$ are related by only one minimal modification). However, to obtain the total contribution to the weights we have, of course, to sum over all allowed $\mathcal{L}$.

The justification for the use of the loop $\mathcal{L}$ is that it allows us to make a systematic correspondence between paths $P_2(s_{1j}, s_{2j})$ which have some part in $V_{j+1} - V_j$ and paths $P_1(s_{1j}, s_{2j})$ which stay in V_j. Let us call $P_2(s_{kj}, s_{k+1j})$ a sub-path of $P_2(s_{1j}, s_{2j})$ in $V_{j+1} -$

V_j, s_{kj} and s_{k+1j} being vertices of $\{s_j\}$. To $P_2(s_{kj}, s_{k+1j})$ we associate a path $P_1(s_{kj}, s_{k+1j})$ in V_j so that $P_1(s_{kj}, s_{k+1j})\ P_2(s_{k+1j}, s_{kj})$ will form a loop $\mathcal{L}_k$. Let us give an ensemble $\{\mathcal{L}_k\}$ of these loops, each being not incident with any another one. Relating the loops $\mathcal{L}_k$ we have paths $P(s_{kj}, s_{k'j})$ in V_j, with $k' \neq k+1$, which are not incident with $\{\mathcal{L}_k\}$ except at s_{kj} and $s_{k'j}$ and which are common to $P_1(s_{1j}, s_{2j})$ and $P_2(s_{1j}, s_{2j})$. It is clear then, that the sets $\{\mathcal{L}_k\}$ allows us to *make a systematic correspondence between all self-avoiding paths* $P_1(s_{1j}, s_{2j})$ *and all self-avoiding paths* $P_2(s_{1j}, s_{2j})$:

$$P_1(s_{1j}, s_{2j}) \overset{\{\mathcal{L}_k\}}{\longleftrightarrow} P_2(s_{1j}, s_{2j}) \tag{52}$$

where $\{\mathcal{L}_k\}$ and the paths $P_1(s_{1j}, s_{2j})$, $P_2(s_{1j}, s_{2j})$ are incident with $\{s_j\}$ at the same vertices.

Now, on each $\mathcal{L}_k$ we can make a cut either in V_j or in $V_{j+1} - V_j$ in order to go from a partition where s_{kj} and s_{k+1j} are disconnected in V_j to partition where they are connected in V_j. However, we see that for each k the cutting of a loop $\mathcal{L}_k$ corresponds to a different minimal modification of the partition $\{s_j\}$. Moreover, each loop $\mathcal{L}_k$ has to be cut in order to have a tree. Therefore, to p loops $\mathcal{L}_k$ in $\{\mathcal{L}_k\}$ we have a set of p cuttings. Here, *requiring only one minimal modification to take place*, we want to cut $P_1(s_{1j}, s_{2j})$ at most once. Suppose this cut takes place on a given loop $\mathcal{L}_k$. Then, we will associate to $P_1(s_{1j}, s_{2j})$ the path $P_2(s_{1j}, s_{2j})$ which differs from $P_1(s_{1j}, s_{2j})$ only on $\mathcal{L}_k$, i.e. *we will have only one* $\mathcal{L}_k$ *in* $\{\mathcal{L}_k\}$. Then, $P_1(s_{1j}, s_{2j})$ will consist of the succession of paths $P(s_{1j}, s_{kj})$ $P_1(s_{kj}, s_{k+1j})\ P(s_{k+1j}, s_{2j})$, all in V_j, see fig. 2. The simplest topology appears when s_{1j} is s_{kj} and s_{2j} is s_{k+1j}, $\mathcal{L}_k$ becoming the loop $\mathcal{L}$ described before. In the following, we will treat this simple case first because the reasoning is almost unchanged passing from $\mathcal{L}$ to $\mathcal{L}_k$. We now turn to the construction of spanning m'-trees in V_{j+1}, and first of spanning trees in V_{j+1}.

We observe that the branches in i) and ii) are exactly the same. The corresponding spanning trees in V_{j+1} only differ in the way $\mathcal{L}$ is cut.

From the spanning trees, m'-trees in V_{j+1} are obtained by cutting off $m'-1$ propagators either on $\mathcal{L}$ or on branches. The rule to be observed is that any sub-tree generated by the cutting should be incident with at least one vertex of the border of V_{j+1} with $G-V_{j+1}$, i.e. a vertex of $\{s_{j+1}\}$. Otherwise, a sub-tree would be isolated from all the others on G and the ensemble of sub-trees could not form a spanning tree on G as they should.

Let us cut branches first. For the reason given above a branch can only be cut when it is incident or connected to a vertex of $\{s_{j+1}\}$. It is clear that branches will be cut in exactly the same way for spanning trees in i) and ii).

Then, we come to the eventual cutting of $\mathcal{L}$, i.e. $P(v_2, v_1)$ or $P(v_1', v_2')$. However, remember that we want that in $\{s_j\}_1$ $P_1(s_{1j}, s_{2j})$ to be cut only once and in $\{s_j\}_2$ not to be cut at all. Therefore, we don't allow $P(v_2, v_1)$ and $P(v_1', v_2')$ to be cut in V_j. However, they can be cut in $V_{j+1} - V_j$. After the branch-cutting has been completed we focus our attention on those branches which are still incident or connected with a vertex of $\{s_{j+1}\}$. Let us call r_k the roots on $\mathcal{L}$ of these particular branches and let us order them along $\mathcal{L}$. It is clear that we can cut $\mathcal{L}$ only once between r_k and r_{k+1} because otherwise the part of $\mathcal{L}$ between two cut propagators, having only branches rooted on it not incident or connected to $\{s_{j+1}\}$, would be disconnected from all other sub-trees on G, which is forbidden.

Moreover, we want the partition $\{s_{j+1}\}$ to be the same for m'-trees constructed from $P(v_2, v_1)$ or $P(v_1', v_2')$. Cutting $\mathcal{L}$ between two roots r_k and r_{k+1} will in general modify $\{s_{j+1}\}$ (except when $\mathcal{L}$ is cut only once) and therefore any path $P(r_k, r_{k+1})$ on $\mathcal{L}$ should be cut or not cut at the same time for the m'-trees generated from $P(v_2, v_1)$ and $P(v_1', v_2')$.

The exception is when only one cut is performed on $\mathcal{L}$, i.e. leaving $P(v_2, v_1)$ and $P(v_1', v_2')$ uncut, because all roots on $\mathcal{L}$ are still connected and in particular the roots r_k. Then, (v_1, v_2) can be anywhere on $P_1(s_{1j}, s_{2j})$ and (v_1', v_2') anywhere on $P_2(s_{2j}, s_{1j})$ in $V_{j+1} - V_j$.

Let us return to the general case when $\mathcal{L}$ is cut more than once. Cutting $\mathcal{L}$ at $(v_1 v_2)$

or $(v'_1 v'_2)$ should not modify $\{s_{j+1}\}$. Therefore, $(v_1 v_2)$ and $(v'_1 v'_2)$ should be on the same path $P(r_{k_1}, r_{k_2})$ on $\mathcal{L}$, r_{k_1} and r_{k_2} being two consecutive roots of type r_k on $\mathcal{L}$. Of course, $P(r_{k_1}, r_{k_2})$ should cross the border of V_j with $V_{j+1} - V_j$ in order to have $(v_1 v_2)$ in V_j and $(v'_1 v'_2)$ in $V_{j+1} - V_j$.

Let us now consider roots on $\mathcal{L}$ of branches in V_j which are incident with or connected to branches in V_j incident with at least one vertex of $\{s_j\}$. Let us call such roots r_b. Again, it is clear that $\mathcal{L}$ can only be cut once between two successive roots r_b and r_{b+1} along $\mathcal{L}$ for the same reason as for the roots r_k. Furthermore, any cut on a path $P(r_b, r_{b+1})$ on $\mathcal{L}$ gives rise to m'-tree in V_{j+1} with m-trees in V_j corresponding to the same partition $\{s_j\}$. Of course, the propagator $(v_1 v_2)$ on $\mathcal{L}$ is on such a path (which itself is on $P(r_{k_1}, r_{k_2})$) which we will call $P(r_{b_1}, r_{b_2})$, i.e. $b_2 = b_1 + 1$, as well as $k_2 = k_1 + 1$. And the propagator $(v'_1 v'_2)$ is on the intersection of $P_2(s_{2j}, s_{1j})$ with $P(r_{k_1}, r_{k_2})$, this intersection being $P_2(s_{2j}, s_{1j})$ if $P(r_{k_1}, r_{k_2})$ contains $P_2(s_{2j}, s_{1j})$ (in which case r_{k_1} and r_{k_2} are in V_j), or a path $P(r_{k_1}, s_{1j})$ on $P_2(s_{2j}, s_{1j})$ if r_{k_1} is in $V_{j+1} - V_j$, or a path $P(s_{2j}, r_{k_2})$ on $P_2(s_{2j}, s_{1j})$ if r_{k_2} is in $V_{j+1} - V_j$. In any case let us call this intersection path P_{int} which of course is always in $V_{j+1} - V_j$. Now, *given* $\mathcal{L}$, we can now write easily the ratio of the weights of m'-trees where P_{int} is cut to those where $P(r_{b_1}, r_{b_2})$ is cut, this is ($\mathcal{L}$ cut more than once, $m' > 1$)

$$\sum_{\ell_2 \subset P_{int}} \bar{\alpha}_{\ell_2} / \sum_{\ell_1 \subset P(r_{b_1}, r_{b_2})} \bar{\alpha}_{\ell_1} \tag{53}$$

which is a remarkably simple expression. In this minimal modification $\{s_j\}_1$ goes to $\{s_j\}_2$ and $\{s_{j+1}\}$ is unchanged. In the case when $m' = 1$, $\mathcal{L}$ is cut only once, the roots r_{k_1} and r_{k_2} are irrelevant to determine where $P_2(s_{2j}, s_{1j})$ should be cut in order to keep $\{s_{j+1}\}$ unchanged because all roots on $\mathcal{L}$ are in any way connected, and P_{int} has to be replaced by $P_2(s_{2j}, s_{1j})$ in the above expression.

We now want to make an important observation, i.e. that in the ratio (53) the only object which may be sensitive to $\{s_{j+1}\}$, $\{s_j\}$ being fixed, is P_{int} through the position of

r_{k_1} or r_{k_2} when r_{k_1} or r_{k_2} are in $V_{j+1} - V_j$. $P(r_{b_1}, r_{b_2})$ being in V_j is insensitive to $\{s_{j+1}\}$, because *once* $\{s_j\}$ *is fixed,* r_{b_1} *and* r_{b_2} *cannot depend on the structure of the* m'*-trees outside* V_j. Or, said otherwise, the structure of m-trees in V_j only depend on $\{s_j\}$.

In what follows, we are going to show that in spite of that, we *can constrain the building of the successive volumes* V_j, $V_{j+1}, \cdots$ *in such a way that the ratio of weights (51) does not depend on a change of* $\{s_{j+1}\}$. Then, if it is finite or infinite for one given $\{s_{j+1}\}$ it stays so for any $\{s_{j+1}\}$. As noted before in the discussion of the weight-ratio (50) this solves immediately our convergence problem for $R_i^m(\{s_j\})$.

Constraint on the construction of the V_j's

Any propagator going out of V_j is relating a vertex of the border of the deformed sphere S_j to a vertex of the border of the deformed sphere S_{j+1} if the vertices at the ends of this propagator are one in V_j and the other one in $V_{j+1} - V_j$. □

The later provision takes into account the possibility for the borders of the deformed S_j and S_{j+1} to coincide on some domain in which case a propagator going out of S_j would also go out of S_{j+1}. This constraint is easy to satisfy because we only need to make the radius of S_{j+1} sufficiently close to that of S_j in order to obey it. It has the following consequence : if a loop $\mathcal{L}$ enters $V_{j+1} - V_j$ at s_{2j} and reenters V_j at s_{1j}, then the part of $\mathcal{L}$ in $V_{j+1} - V_j$ is a path

$$(s_{2j}\ s_{2j+1})\ P(s_{2j+1}, s_{1j+1})(s_{2j+1}, s_{1j}) \tag{54}$$

where $(s_{2j}\ s_{2j+1})$ and $(s_{1j+1}\ s_{1j})$ are two propagators relating vertices on the border of V_j and V_{j+1}. Furthermore any vertex of $P(s_{2j+1}, s_{1j+1})$ not on the border of V_{j+1} is related to vertices of V_j through paths on $P(s_{2j+1}, s_{1j+1})$ going to vertices on the border of V_{j+1}, see fig. 3.

An immediate consequence of the structure of $\mathcal{L}$ in $V_{j+1} - V_j$ as shown in (54) is that s_{1j+1} is r_{k_1} or s_{2j+1} is r_{k_2} because being in $\{s_{j+1}\}$ and on $\mathcal{L}$ they are roots of branches

incident with $\{s_{j+1}\}$, these branches being restricted to one vertex. Then, P_{int} is simply the propagator $(s_{1j+1}\ s_{1j})$ or the propagator $(s_{2j}\ s_{2j+1})$ depending on which part of $\mathcal{L}$ we choose $P(r_{k_1}, r_{k_2})$ to be. *However, in any case,* P_{int} *depends only on* $\mathcal{L}$ *and no more on* $\{s_{j+1}\}$*, because the vertices* s_{1j+1} *and* s_{2j+1} *will not move on* $\mathcal{L}$ *as we change* $\{s_{j+1}\}$. Then, the ratio (53) will depend on $\mathcal{L}$ and not $\{s_{j+1}\}$ and the ratio of the weights for two partitions $\{s\}_1$ and $\{s_j\}_2$ related by a minimal modification will be, *summing over all* $\mathcal{L}$ *going out of* V_j *at* s_{2j} *and reentering* V_j *at* s_{1j}. ($\mathcal{L}$ cut more than once, $m' > 1$),

$$W^{m'}_{m_1-1}(\{s_j\}_2, \{s_{j+1}\})/W^{m'}_{m_1}(\{s_j\}_1, \{s_{j+1}\}) =$$

$$\bar{\alpha}_{P_{int}} \sum_{T^{m_1-1}} T^{m_1-1}(\bar{i}, \{s_j\}_2)/ \sum_{T^{m_1-1}} \Big(\sum_{\ell_1 \subset P(r_{b_1}, r_{b_2})} \bar{\alpha}_{\ell_1} \Big) T^{m_1-1}(\bar{i}, \{s_j\}_2) \tag{55}$$

where $T^{m_1-1}(\bar{i}, \{s_j\}_2)$ is for the weight of spanning (m_1-1)-trees in V_j with partition $\{s_j\}_2$ and $\bar{\alpha}_{P_{int}}$ is for the $\bar{\alpha}_\ell$ of the propagator $(s_{1j+1}\ s_{1j})$ or $(s_{2j}\ s_{2j+1})$. The sum over the spanning (m_1-1)-trees is provided by cutting in all possible ways compatible with $\{s_j\}_2$ m_1-2 propagators from all spanning trees in V_j, not going through the propagator i, and obtained from all possible paths $P_2(s_{2j}, s_{1j})$. Looking at the weight structure in (49.b) we see that the contribution to weights coming from sub-trees in $V_{j+1} - V_j$, $\sum_{T^n} T^n(\{s'_j\}_1, \{s_{j+1}\})$, being the same ones for $W^{m'}_{m_1-1}(\{s_j\}_2, \{s_{j+1}\})$ and $W^{m'}_{m_1}(\{s_j\}_1, \{s_{j+1}\})$, and being factorized, cancels out in (55). The main feature of the expression on the right-hand side of (55) is that it does not depend on $\{s_{j+1}\}$, which is what we were looking for.

We now have to establish the same property for the contribution coming from spanning m'-trees in V_{j+1} where $\mathcal{L}$ *is only cut once.* For this case we need to separate two classes of m'-trees

a) those m'-trees where for $\{s_j\}_2$, $\mathcal{L}$ is cut on P_{int}, i.e. on the propagator $(s_{2j}\ s_{2j+1})$ or $(s_{1j+1}\ s_{1j})$.

b) those m'-trees where for $\{s_j\}_2$, $\mathcal{L}$ is cut on the complement of P_{int} on $P_2(s_{2j}, s_{1j})$, which we will call P^{comp}_{int}, but uncut on P_{int}.

For the m'-trees in a) the reasoning is the same as for the m'-trees where $\mathcal{L}$ is cut more than once and the result (55) is valid for them too. For the m'-trees in b), changing $\{s_{j+1}\}$, *we cannot obtain m'-trees in which $\mathcal{L}$ is cut more than once*, because $P(r_k, r_{k_2})$ would stay uncut for $\{s_j\}_2$ - corresponding m'-trees, which is forbidden (i.e., for $\mathcal{L}$ cut more than once, $P(r_{k_1}, r_{k_2})$ has to be cut for $\{s_j\}_1$ in V_j and for $\{s_j\}_2$ in $V_{j+1} - V_j$, i.e. on P_{int}, in order not to change $\{s_{j+1}\}$ passing from $\{s_j\}_1$ to $\{s_j\}_2$). We remind the reader that when $\mathcal{L}$ is cut only once all roots r_k on $\mathcal{L}$ are connected. Then, for the m'-trees in b) changing $\{s_{j+1}\}$ always keeps all roots r_k on $\mathcal{L}$ connected because $\mathcal{L}$ stays cut only once. Let us then take *all the loops $\mathcal{L}$ with the same P_{int}^{comp}*, for these the contribution of the weights will be such that ($\mathcal{L}$ cut once, $m' \geq 1$)

$$W_{m_1-1}^{m'}(\{s_j\}_2, \{s_{j+1}\}) / W_{m_1}^{m'}(\{s_j\}_1, \{s_{j+1}\}) =$$

$$\Big(\sum_{\ell_2 \subset P_{int}^{comp}} \bar{\alpha}_{\ell_2}\Big) \sum_{T^{m_1-1}} T^{m_1-1}(\bar{\imath}, \{s_j\}_2) / \sum_{T^{m_1-1}} \Big(\sum_{\ell_1 \subset P(r_{b_1}, r_{b_2})} \bar{\alpha}_{\ell_1}\Big) T^{m_1-1}(\bar{\imath}, \{s_j\}_2) \qquad (56)$$

which, again, is independent of $\{s_{j+1}\}$.

Now, because each contribution to the weight $W_{m_1}^{m'}(\{s_j\}_1, \{s_{j+1}\})$ is multiplied by a factor independent of $\{s_{j+1}\}$ when the corresponding contribution to $W_{m_1-1}^{m'}(\{s_j\}_2, \{s_{j+1}\})$ is taken, we have the following theorem :

Theorem 2a

Let us consider a minimal modification of $\{s_j\}$, $\{s_j\}_1 \to \{s_j\}_2$ where $\{s_{j+1}\}$ remains unchanged. In $\{s_j\}_2$, s_{1j} and s_{2j}, vertices of $\{s_j\}$ are connected by a self-avoiding path $P_1(s_{1j}, s_{2j})$ in V_j. In $\{s_j\}_1$, $P_1(s_{1j}, s_{2j})$ is cut once. The m'-trees in V_{j+1} corresponding to $\{s_j\}_2$ are obtained from $P_1(s_{1j}, s_{2j})$. The m'-trees in V_{j+1} corresponding to $\{s_j\}_1$ are obtained from a self-avoiding path $P_2(s_{2j}, s_{1j})$ with all propagators in $V_{j+1} - V_j$, its only vertices on the border of $V_{j+1} - V_j$ with V_j being s_{1j} and s_{2j}.

Then, the ratio of the sum of weights of m'-trees corresponding to $\{s_j\}_2$ to the sum of the weights of m'-trees corresponding to $\{s_j\}_1$ is independent of $\{s_{j+1}\}$. $\square$

We have examined so far the simplest case where the loop $\mathcal{L}_k$ was taken as the loop $\mathcal{L}$ going out of V_j at s_{2j} and in V_j at s_{1j}. In general, as discussed earlier, a path $P_2(s_{1j}, s_{2j})$ may have some parts $P_2(s_{kj}, s_{k+1j})$ in $V_{j+1} - V_j$. Confining ourselves to the minimal modification of $\{s_j\}$, only one such $P_2(s_{kj}, s_{k+1j})$ or one $\mathcal{L}_k$ is relevant.

We can repeat the reasoning followed with the loop $\mathcal{L}$ for the loop $\mathcal{L}_k$, the only change being that two self-avoiding paths in V_j, $P(s_{1j}, s_{kj})$ and $P(s_{k+1j}, s_{2j})$ will be rooted at s_{kj} and s_{k+1j} respectively on $\mathcal{L}_k$. Then, we have to divide the paths $P_1(s_{1j}, s_{2j})$ into two classes in order to avoid a double-counting :

a) those which are incident with a vertex of $\{s_j\}$ other than s_{1j} and s_{2j} only once

b) those which are incident with at least two vertices s_{kj} and $s_{k'j}$ of $\{s_j\}$.

The paths a) will be associated to the loop $\mathcal{L}$. The paths b) will be associated to a loop $\mathcal{L}_k$ with $s_{k'j}$ being then noted s_{k+1j}. Of course, in this case $P_1(s_{1j}, s_{2j})$ can be incident with other vertices of $\{s_j\}$ as well. We only need to exhaust all pairs of vertices s_{kj}, $s_{k'j}$ in order to form all possible loops $\mathcal{L}_k$. In this way, all different paths $P_1(s_{1j}, s_{2j})$ are taken into account only once. Theorem 2a applied to each pair s_{kj}, $s_{k'j}$ replacing s_{1j} and s_{2j} then provides the independence of the weight ratio (51) on $\{s_{j+1}\}$.

The only thing which is left to prove is that, indeed, any partition $\{s_j\}_2$ can be obtained from another one $\{s_j\}_1$ by a series of minimal modifications. So let us consider the respective situation of two vertices s_{1j} and s_{2j} in $\{s_j\}_1$ and in $\{s_j\}_2$. Suppose that they are unconnected in $\{s_j\}_1$ and connected in $\{s_j\}_2$. Then, it is easy to see that a minimal modification will allow to disconnect them, passing from $\{s_j\}_2$ to $\{s_j\}_3$, leaving weight-ratios corresponding to those in (51) independent of $\{s_{j+1}\}$. Then, we will consider the respective situation of any two other s_j's and repeat the operation until we obtain $\{s_j\}_1$, having always weight-ratios insensitive to $\{s_{j+1}\}$.

Let us make also a remark about the topology of G in V_{j+1}. G can be disconnected

into several pieces in V_{j+1}, although G itself is taken to be connected and even 1-line and 1-vertex irreducible. When G is multiply connected in V_{j+1} we consider each connected piece separately for the construction of spanning trees on them and the eventual cutting of propagators. The resulting weight for m'-trees will simply be the product of the weights of all connected pieces of G in V_{j+1}. Of course, in this case $m' > 1$ but the above reasoning is essentially unchanged. We then are able to write the following theorem :

Theorem 2b

For any two different partition $\{s_j\}_1$ and $\{s_j\}_2$ of vertices s_j on the border of V_j with $V_{j+1} - V_j$ the ratio of weights (51) is independent of $\{s_{j+1}\}$ if we impose the constraint on the construction of the V_j's described earlier and which can always be satisfied. It follows that the ratio (50)

$$\sum_{\{s_j\}_M} W^{m'}_{m_M}\left(\{s_j\}_M, \{s_{j+1}\}\right) / \sum_{\{s_j\}} W^{m'}_{m}\left(\{s_j\}, \{s_{j+1}\}\right)$$

is also independent of $\{s_{j+1}\}$. Therefore, when this ratio is infinite it is so for all $\{s_{j+1}\}$ and $R^{m'}_i(\{s_{j+1}\})$ takes a unique value $R^m_i(\{s_{j+1}\}_M)$. When this ratio is finite, it also stays finite for all $\{s_{j+1}\}$ and therefore the convergence condition (41) is satisfied. $\square$

In order to have $R^m_i(\{s_j\})$ to converge as $j \to \infty$, we also need to have this ratio finite for some finite value of j. This condition is naturally satisfied if we impose the constraint that V_1 should only contain a finite number of propagators. This condition, of course, can always be satisfied by choosing a sufficiently small radius for the sphere S_1.

C - *Extension of the convergence proof to trees*

We now want to extend the inequality (36) of theorem 1 to the spanning trees in V_j and V_{j+1}. For that purpose let us define a (spanning) multiple tree (or m'-tree) $T^{m'}(i, \{s_{j+1}\})$ contained in V_{j+1}, one component of which is going through i. $\{s_{j+1}\}$ stands for a partition of all border-vertices of V_{j+1} with $G - V_{j+1}$ with which $T^{m'}(i, \{s_{j+1}\})$ is incident. As for

paths, $T^{m'}(\bar{i}, \{s_{j+1}\})$ has the same properties as $T^{m'}(i, \{s_{j+1}\})$, except that it does not go through i but is still incident with v_i through one to its component-tree. In the following, in order to have a simpler notation we will imply that *a summation over all T's of the kind is made whenever the symbol T appears.* Then, we can write, $\{s_j\}$ being some partition of vertices of the border of V_j with $V_{j+1} - V_j$ and $\{s_{j+1}\}$ a corresponding partition for V_{j+1}, ($T^m(i, \{s_j\})$ is a m-tree contained in V_j, $T(\{s_j^t\})$ is a tree in $V_{j+1} - V_j$ with no vertex on the border of V_{j+1} and incident with the border of V_j at $\{s_j^t\}$, $T(\{s_j^u\}\,\{s_{j+1}^u\})$ is a tree in $V_{j+1} - V_j$ incident with the borders of V_j and V_{j+1} at $\{s_j^u\}$ and $\{s_{j+1}^u\}$ respectively, and $T(\{s_{j+1}^v\})$ is a tree in $V_{j+1} - V_j$ not incident with the border of V_j but incident with the border of V_{j+1} at $\{s_{j+1}^v\}$)

$$T^{m'}(i, \{s_{j+1}\}) = \sum_{m=1}^{M_j} \sum_{\{s_j\}} T^m(i, \{s_j\}).$$

$$\sum_{\{\{s_j^t\}\}} \prod_{t=1}^{x} T(\{s_j^t\}) \sum_{\{\{s_j^u\},\{s_{j+1}^u\}\}} \prod_{u=1}^{y} T(\{s_j^u\}, \{s_{j+1}^u\}) \sum_{\{\{s_{j+1}^v\}\}} \prod_{v=1}^{z} T(\{s_{j+1}^v\}) \tag{57}$$

($\{\{s_j^u\}, \{s_{j+1}^u\}\}$ is the ensemble of border-vertices of the y trees $T(\{s_j^u\}, \{s_{j+1}^u\})$, $\{\{s^t\}\}$ is the ensemble of the border-vertices of the x trees $T(\{s_j^t\})$ and $\{\{s_{j+1}^v\}\}$ is the ensemble of border-vertices of the z trees $T(\{s_{j+1}^v\})$) for x, y, $z \geq 1$ and with

$$y + z \geq m' \geq z \tag{58}$$

When $x = 0$, $y = 0$ or $z = 0$ the corresponding product in (57) is equal to one (no tree). The relation (51) can be explained by saying that $y + z$ is the maximum number of components of $T^{m'}(i, \{s_{j+1}\})$ and z the minimum number of its components. A tree $T(\{s_j^t\})$ in $V_{j+1} - V_j$ should be connected to at least one tree in V_j, giving

$$\forall\ t, \qquad \{s_j\} \cap \{s_j^t\} \neq \phi \tag{59}$$

If $\{s_j^{m_c}\}$, $(m_c = 1, \cdots, m)$, denotes the set of border-vertices of V_j of one component-tree of $T^m(i, \{s_j\})$, i.e. if $\{s_j\} = \{\{s_j^{m_c}\}\}$, then, avoiding loops,

$$\{s_j^t\} \cap \{s_j^{m_c}\} = \text{at most one } s_j \tag{60.a}$$

$$\{s_j^u\} \cap \{s_j^{m_c}\} = \text{at most one } s_j \tag{60.b}$$

Any component-tree of $T^m(i, \{s_j\})$ should be connected to at least one tree in $V_{j+1} - V_j$ giving

$$\forall\ m, \quad [\{\{s_j^t\}\} \cup \{\{s_j^u\}\}] \cap \{s_j^{m_c}\} \neq \phi \tag{60.c}$$

The relations (60) insure that a component-tree of $T^m(i, \{s_j\})$ in V_j should be incident with at least one component tree in $V_{j+1} - V_j$ through at most one vertex. Indeed, (59) and (60) fix the topology of contact between the component-trees in V_j and $V_{j+1} - V_j$ in order to leave no component-tree isolated and in a way that avoids the formation of any loop. Let us denote $\{\{s_j^t\}\} \cup \{\{s_j^u\}\}$ by $\{s'_j\}$ which will be the ensemble of the vertices on the border of V_j of the trees in $V_{j+1} - V_j$. Then, it is clear that once $\{s_j\}$ and $\{s'_j\}$ have been fixed, the summation over trees in V_j and $V_{j+1} - V_j$ is factorizable because no interaction occurs between the trees in V_j and those in $V_{j+1} - V_j$ (apart from their contact at the common border of V_j and $V_{j+1} - V_j$). Note even that once $\{s_j\}$ is fixed the summation over all $T^m(i, \{s_j\})$ (or over all $T^m(\bar{i}, \{s_j\})$) does not depend on $\{s'_j\}$. Consequently, the sum over all n-trees in $V_{j+1} - V_j$, *including the sum over* $\{s'_j\}$, can be factorized out.

Again, in a way analogous to that of paths let us define

$$R_i^m(\{s_j\}) = \sum_{T^m} T^m(i, \{s_j\}) / \sum_{T^m} T^m(\bar{i}, \{s_j\}) \tag{61}$$

for which we demonstrate as for paths the following

Theorem 3

$$R_i^m(\{s_j\})_{min} < R_i^{m'}(\{s_{j+1}\}) < R_i^m(\{s_j\})_{max} \tag{62}$$

where $R_i^m(\{s_j\})_{min}$ and $R_i^m(\{s_j\})_{max}$ are respectively the minimum and the maximum

value of $R_i^m(\{s_j\})$ for all m's or $R_i^{m'}(\{s_{j+1}\})$ has converged to either $R_i^m(\{s_j\})_{min}$ or $R_i^m(\{s_j\})_{max}$.

Proof

It is clear, using (61) in (57) that $R_i^{m'}(\{s_{j+1}\})$ is a mean-value of $R_i^m(\{s_j\})$ considered as a function of m, $\{s_j\}$ and $\{\{s_j\}\}$, $\{\{s'_j\}\}$. However, due to the factorization property mentioned above (once $\{s_j\}$ is fixed, $\{s'_j\}$ does not have an influence over the m-trees in V_j), the functional dependence of $R_i^m(\{s_j\})$ is indeed restricted to m and $\{s_j\}$. Moreover, theorem 2b either excludes the limiting cases where the inequalities become equalities or makes $R_i^{m'}(\{s_{j+1}\})$ equal to $R_i^m(\{s_j\})_{min}$ or $R_i^m(\{s_j\})_{max}$ for any $\{s_{j+1}\}$. It therefore follows that either (62) is true or $R_i^{m'}(\{s_{j+1}\})$ has converged. □

<u>Remark</u>

We note that $R_i^m(\{s_j\})$ (and $T^m(i,\{s_j\})$ is a function of a partition $\{s_j\}$ which covers any part of the border of V_j on the surface of the deformed S_j independently of the partition $\{s_{j+1}\}$ of $T^{m\prime}(i,\{s_{j+1}\})$. Therefore, the sum $\sum\limits_{\{s_j\}}$ in (57) is a sum over the *whole* part of border of V_j on the deformed surface S_j. This is in contrast with the corresponding sums $\sum\limits_{s_j^0}$ or $\sum\limits_{\{s_j^l\}}$ in (34) for paths which may cover only part of the border of V_j on the deformed surface S_j, depending on s_{j+1} in $P(i,s_{j+1})$. Therefore, in the case of trees, the topology of $V_{j+1}-V_j$ does not intervene to possibly limit the range of $\{s_j\}$. Hence, we have a unique value for $R_i^m(\{s_j\})$ and this solves problem b) of the preceding section. The repeated use of (62) will make all $R_i^m(\{s_j\})$ converge towards the same value R_i^∞.

5. The factorization of trees on G

We now proceed to the proof of the relation (24), the crucial factorization property of spanning trees on G.

Let us consider a volume V_j with $j\to\infty$, and a partition of its border-vertices $\{s_j\}$ and

let us denote by R_i^∞ the common value to which all $R_i^m(\{s_j\})$ tend as $j \to \infty$. Recalling (16) and (17) we have, T^n being an n-tree on $G - V_j$, (a sum over m is implied in the sum over all $\{s_j\}$ and a sum over n is implied in the sum over all $\{s'_j\}$ compatible with $\{s_j\}$)

$$\bar{a}_i = \Delta(\bar{\alpha}) \sum_{\mathcal{T} \supset i} \prod_{l \in \mathcal{T}} \bar{\alpha}_l^{-1}$$

$$= \Delta(\bar{\alpha}) \sum_{\{s_j\}, T^m} T^m(i, \{s_j\}) \sum_{\{s'_j\}, T^n} T^n(\{s'_j\}) \tag{63.a}$$

$$\bar{b}_i = \bar{\alpha}_i^{-1} \Delta(\bar{\alpha}) \sum_{\mathcal{T} \not\supset i} \prod_{l \in \mathcal{T}} \bar{\alpha}_l^{-1}$$

$$= \bar{\alpha}_i^{-1} \Delta(\bar{\alpha}) \sum_{\{s_j\}, T^m} T^m(\bar{i}\{s_j\}) \sum_{\{s'_j\}, T^n} T^n(\{s'_j\}) \tag{63.b}$$

$\{s'_j\}$ is a subset of the vertices of V_j belonging to $T^n(\{s'_j\})$, and $T^n \cup T^m$ form a spanning tree of G. Notice that the sets $\{s'_j\}$ and $\{s_j\}$ are in general different, a branch of T^m can end at one s_j without being incident at that s_j with one branch of T^n. For any given m, $\{s_j\}$, $T^m(i, \{s_j\})$ can be replaced by $R_i^\infty \; T^m(\bar{i}, \{s_j\})$ and we therefore get

$$\bar{a}_i / \bar{b}_i = R_i^\infty \bar{\alpha}_i \quad . \tag{64}$$

Remember that by a given $\{s_j\}$ we mean a given $\{\{s_j^{m_c}\}\}$ where $\{s_j^{m_c}\}$ is a set of s_j's belonging to the same component-tree of T^m. In the same way $\{s'_j\}$ means $\{\{s_j^{\prime n_c}\}\}$ where $\{s_j^{\prime n_c}\}$ belongs to one component-tree of T_n.

Looking now at (22) and (23) we can write, using the same notations,

$$\bar{d}_{i,k} = \sum_{\{s_j\}, T^m} s_{\mathcal{C}_k} \; \nu^{-1}(\mathcal{C}_k) \; T^m(i, \{s_j\}) \sum_{\{s'_j\}, T_k^n} T_k^n(\{s'_j\}) \tag{65.a}$$

$$\bar{e}_{i,k} = \bar{\alpha}_i^{-1} \sum_{\{s_j\}, T^m} s_{\mathcal{C}_k} \; \nu^{-1}(\mathcal{C}_k^i) \; T^m(\bar{i}, \{s_j\}) \sum_{\{s'_j\}, T_k^n} T_k^n(\{s'_j\}) \tag{65.b}$$

where T_k^n is a n-tree belonging to $G - V_j$ and going through the propagator k which is assumed to be outside V_j.

We see that, compared to (63), the expressions (65) have exactly the same structure except for the factors $s_{\mathcal{C}_k}\ \nu^{-1}(\mathcal{C}_k)$ and $s_{\mathcal{C}_k}\ \nu^{-1}(\mathcal{C}_k^i)$. We now want to demonstrate that, indeed,

$$[s_{\mathcal{C}_k}\ \nu^{-1}(\mathcal{C}_k)]/[s_{\mathcal{C}_k^i}\ \nu^{-1}({\mathcal{C}_k}^i)] = 1 + \varepsilon \tag{66}$$

where ε is infinitesimal. Let us recall that $\nu(\mathcal{C}_k)$ counts the number of propagators on the surface $\mathcal{S}_k(\mathcal{C}_k)$ cutting G into two disjoint pieces $G_1(\mathcal{C}_k)$ and $G_2(\mathcal{C}_k)$, the same being true for $\nu(\mathcal{C}_k^i)$ replacing $\mathcal{C}_k$ by $\mathcal{C}_k^i$.

Let us take $j = 1$ for the expressions (65) in order to have a finite number of propagator in V_1. First, it is obvious that if $\mathcal{S}_k$ does not cut through V_1 it will cut only the T^n's which are the same n-trees in (65.a) and (65.b) for a given $\{s_j\}$ and we will have $\varepsilon = 0$ in (66). Now, if j grows, $\mathcal{S}_k$ will remain identical in (65.a) and (65.b) because $\mathcal{C}_k$ and $\mathcal{C}_k^i$ do not depend on j, i.e. on the decomposition of a spanning tree in G into a m-tree in V_j and a n-tree in $G - V_j$. So, ε will remain equal to zero as $j \to \infty$.

The non-trivial case occurs when $\mathcal{S}_k$ cuts through V_1. Then, it could be that when $j \to \infty$ the number of propagators cut by $\mathcal{S}_k$ in V_j remains finite. This could arise when there are domains in the deformed sphere S_j which are empty of propagators and of a size large enough so that when $\mathcal{S}_k$ cuts through them it will contain only a finite number of propagators in V_j. An example of this situation is provided when G has a topology such that it consists of infinite ladders (which may join and separate themselves creating a sort of effective field theory of Reggeons).

To study the situation where S_k cuts through V_1 let us consider, for a given spanning tree $\mathcal{T}$ on G, the sum

$$\Sigma_{\mathcal{T}}(G) = \sum_{k \in \mathcal{T}} \bar{\alpha}_k\ s_{\mathcal{C}_k}\ \nu^{-1}(\mathcal{C}_k) \tag{67}$$

and a similar sum for the part of $\mathcal{T}$ in V_1, $\mathcal{T}_1$

$$\Sigma_{\mathcal{T}}(V_1) = \sum_{k\in\mathcal{T}_1} \bar{\alpha}_k \; s_{\mathcal{C}_k}\nu^{-1}(\mathcal{C}_k)$$
$$=< \bar{\alpha}_k \; s_{\mathcal{C}_k} >_{\mathcal{T}_1} \sum_{k\in\mathcal{T}_1} \nu^{-1}(\mathcal{C}_k) \tag{68}$$

where $< \bar{\alpha}_k \; s_{\mathcal{C}_k} >_{\mathcal{T}_1}$ is the mean-value of $\bar{\alpha}_k \; s_{\mathcal{C}_k}$ for all propagators k in V_1 belonging to $\mathcal{T}$. The maximum value of $\sum_{\mathcal{T}}(V_1)$ is obtained when $\nu(\mathcal{C}_k)$ for all k's belonging to $\mathcal{T}_1$ is constant, meaning that the corresponding S_k's each contain a finite number of propagators. Then, $\sum_{\mathcal{T}}(V_1)$ is equal to $< \bar{\alpha}_k \; s_{\mathcal{C}_k} >_{\mathcal{T}_1}$ multiplied by some constant.

We can express $\sum_{\mathcal{T}}(G)$ in the same way, writing

$$\Sigma_{\mathcal{T}}(G) =< \bar{\alpha}_k \; s_{\mathcal{C}_k} >_{\mathcal{T}} \sum_{k\in\mathcal{T}} \nu^{-1}(\mathcal{C}_k) \tag{69}$$

where $< \bar{\alpha}_k \; s_{\mathcal{C}_k} >_{\mathcal{T}}$ is the mean-value of $\bar{\alpha}_k \; s_{\mathcal{C}_k}$ over $\mathcal{T}$. Now, we can calculate a lower bound for $\sum\limits_{k\in\mathcal{T}}\nu^{-1}(\mathcal{C}_k)$.

This comes from the fact that every propagator in S_k has to be incident with a vertex of the sub-trees in $G_1(\mathcal{C}_k)$ and $G_2(\mathcal{C}_k)$, the parts of G separated by S_k. If a branch consisting of N vertices (and $N-1$ proapagators) is separated in a ϕ^n field theory, the number of propagators cut is $(n-2)N+1$ which represents *the maximum number of propagators cut for a tree with N vertices* (and of course $N-1$ propagators). The number of propagators in $G_1(\mathcal{C}_k)$, for example can be taken to vary from zero to $I-L$. Thus, the following inequality follows

$$\sum_{k\in\mathcal{T}} \nu^{-1}(\mathcal{C}_k) > \frac{1}{n-2}\int_1^{I-L+1} dN/(N+1) = (1/(n-2))\mathrm{Log}(I-L+1) \tag{70}$$

with $(1/(n-2))\mathrm{Log}(I-L+1)$ tending to infinity as $I\to\infty$. Therefore, if the ratio

$$< \bar{\alpha}_k \; s_{\mathcal{C}_k} >_{\mathcal{T}_1} / < \bar{\alpha}_k \; s_{\mathcal{C}_k} >_{\mathcal{T}} \tag{71}$$

stays finite, the ratio

$$\Sigma_{\mathcal{T}}(G)/\Sigma_{\mathcal{T}}(V_1) \tag{72}$$

is infinite as $I \to \infty$ and the contribution to (65.c) and (65.b) coming from S_k's cutting through V_1 can be neglected. It follows, then, that (66) will be true.

Let us now remark that (71) may be infinite in the case where some internal lines incident with vertices in V_1 carry a momentum infinite with respect to the momenta of external lines incident with $G-V_1$. Then, S_k may cut V_1 in such a way as to separate such lines, giving an infinite $s_{\mathcal{C}_k}$, while a cancellation occurs between infinite momenta when S_k does not cut through V_1. Note however that if, although infinite, (71) is equal to

$$\varepsilon_1 \mathrm{Log}(I-L) \tag{73}$$

with $\varepsilon_1 \to 0$ as $I \to \infty$, the conclusion reached above, i.e. that $\sum_{\mathcal{T}}(V_1)$ can be neglected in front of $\sum_{\mathcal{T}}(G)$, is still valid.

Finally, considering the ratio

$$\Sigma_{\mathcal{T}}(G)/\Sigma_{\mathcal{T}}(V_j) \tag{74}$$

where $\sum_{\mathcal{T}}(V_j)$ is the sum of $\bar{\alpha}_k\ s_{\mathcal{C}_k}\nu^{-1}(\mathcal{C}_k)$ for k's belonging to the part of $\mathcal{T}$ in V_j, we see that we have the inequality

$$\Sigma_{\mathcal{T}}(G)/\Sigma_{\mathcal{T}}(V_j) > \left[(n-2)^{-1}\mathrm{Log}(I-L)/(I-L)_{V_j}\right] < \bar{\alpha}_k\ s_{\mathcal{C}_k} >_{\mathcal{T}}\ /< \bar{\alpha}_k\ s_{\mathcal{C}_k} >_{\mathcal{T}_j} \tag{75}$$

if $(I-L)_{V_j}$ is the number of lines a spanning tree in V_j (a m-tree in V_j has even less lines than $(I-L)_{V_j}$) and $< \bar{\alpha}_k\ s_{\mathcal{C}_k} >_{\mathcal{T}}$ the mean-value of $\bar{\alpha}_k\ s_{\mathcal{C}_k}$ over $\mathcal{T}_j$, the m-tree part of $\mathcal{T}$ on V_j.

Provided the ratio on the right of (75) is infinite we can neglect in (20) and (21) the sum over k, k belonging to V_j, even as $(I-L)_{V_j}$ goes to infinity, as was claimed in section 3.

Then, $T^m(i, \{s_j\})$ can be replaced by $R_i^\infty \, T^m(\bar{i}, \{s_j\})$ in (65.a) and (65.b) as in (63.a) and (63.b) with the result

$$\bar{d}_{i,k}/\bar{e}_{i,k} = R_i^\infty \bar{\alpha}_0 \tag{76}$$

which was sought after. This entails that (24) and thereby (23) are verified. As said in section 2 this, in turn, makes $Q_G(P_v, \{\bar{\alpha}_i\})$ insensitive to the replacement $\bar{\alpha}_i \to \bar{\infty}$ and a unique $\bar{\alpha}$ can be used to evaluate F_G in a super-renormalizable scalar field theory.

6. Conclusion

Our initial aim was to put the Gaussian representation for propagators on a firm footing. Using the well-known α-representation we are able to prove that the parameter α which measures the inverse of the variance of that Gaussian can be taken everywhere equal to some unique $\bar{\alpha} = O(1/I)$ where I is the number of internal lines of a Feynman graph G. We did this for a super-renormalizable scalar field theory, although we expect the same result to hold for renormalizable theories as well. But, what is more interesting even, is that we were obliged during the derivation to prove a factorization property of spanning trees on G, i.e., we can sum over all graphs in a volume V_j, and if $j \to \infty$, the structure of the trees outside V_j is independent of the structure of the same trees in V_1, a subpart of V_j. We can however imagine that V_j itself is an infinitesimal volume relative to the whole volume of G. Then, we can interpret our factorization of trees as the factorization of local sums defined on trees. In other words, *trees on a Feynman graph can be used to define a functional integral.* In fact, we assumed such a functional property in our first attempt [7] to derive the relation (13).

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Figure Captions

Fig. 1 We illustrate the case (see eq. (38)) where $m' = 3$, $m = 3$, i.e. there are 3 paths in V_{j+1} and 3 paths in V_j with $t_j = 1$, $u_j = 3$ and $v_{j+1} = 1$. $B(V_{j+1})$ and $B(V_j)$ are respectively the borders of V_{j+1} and V_j.

The path $P(\bar{i}, s_j^0)$ relates one end of the propagator i to s_j^0 without going through i. Paths $P(s_j^{2\ell-1}, s_j^{2\ell})$ corresponding to $\ell = 1$, relating s_j^1 to s_j^2, and to $\ell = 2$, relating s_j^3 to s_j^4 are shown, together with three paths $P(s_j^{\ell u}, s_{j+1}^{\ell u})$ with $u = 1, 2, 3$ and one path $P(s_{j+1}^{2\ell v-2}, s_{j+1}^{2\ell v-1})$ with $v = 1$.

Fig. 2 The case where one loop $\mathcal{L}_k$ is present is depicted. The path $P_1(s_{1j}, s_{2j})$ in V_j consists of the successive paths $P(s_{1j}, s_{kj})\ P_1(s_{kj}, s_{k+1j})\ P(s_{k+1j}, s_{2j})$. The propagator $(v_1 v_2)$ is shown on $P_1(s_{kj}, s_{k+1j})$. The path $P_2(s_{1j}, s_{2j})$ differs from $P_1(s_{1j}, s_{2j})$ by the path $P_2(s_{kj}, s_{k+1j}$ in $V_{j+1} - V_j$ on which the propagator $(v_1' v_2')$ is shown. The loop $\mathcal{L}_k$ is formed by the succession of paths $P_1(s_{kj}, s_{k+1j})\ P_2(s_{k+1j}, s_{kj})$ where the latter is the reverse path of $P_2(s_{kj}, s_{k+1j})$.

Fig. 3 The constraint that any propagator stemming out of V_j should relate the part of the borders of V_j and V_{j+1} (on the deformed spheres S_{j+1} and S_j respectively) has been imposed, if this propagator is in V_{j+1}. The path in V_j $P_1(s_{1j}, s_{2j})$ is shown (here $\mathcal{L}_k$ is simply $\mathcal{L}$). In $V_{j+1} - V_j$, the path $P_2(s_{1j}, s_{2j})$ is shown by a thick line. Depicted are the propagators $(s_{1j}\ s_{1j+1})$ and $(s_{2j}\ s_{2j+1})$.

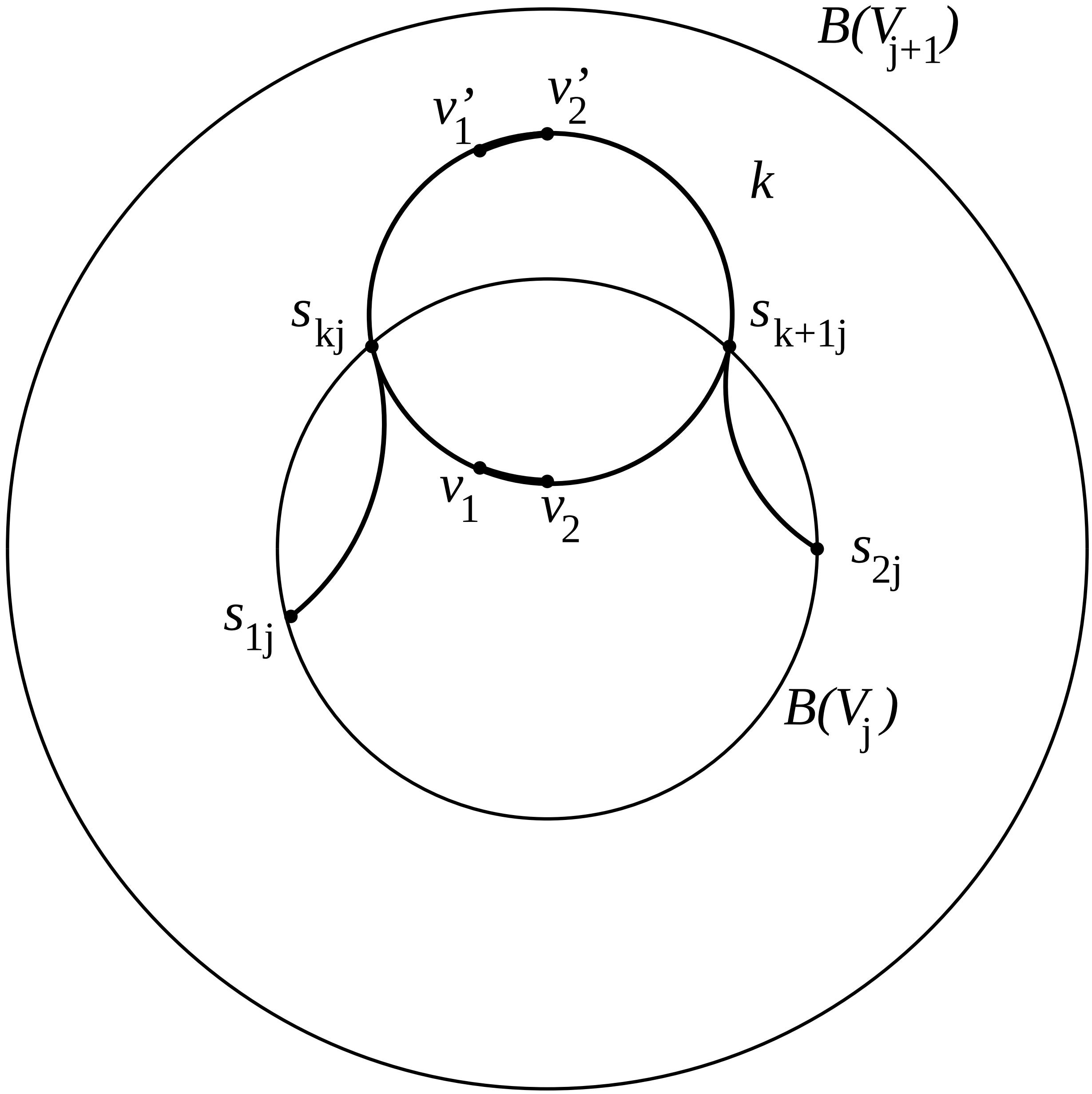

Fig. 2

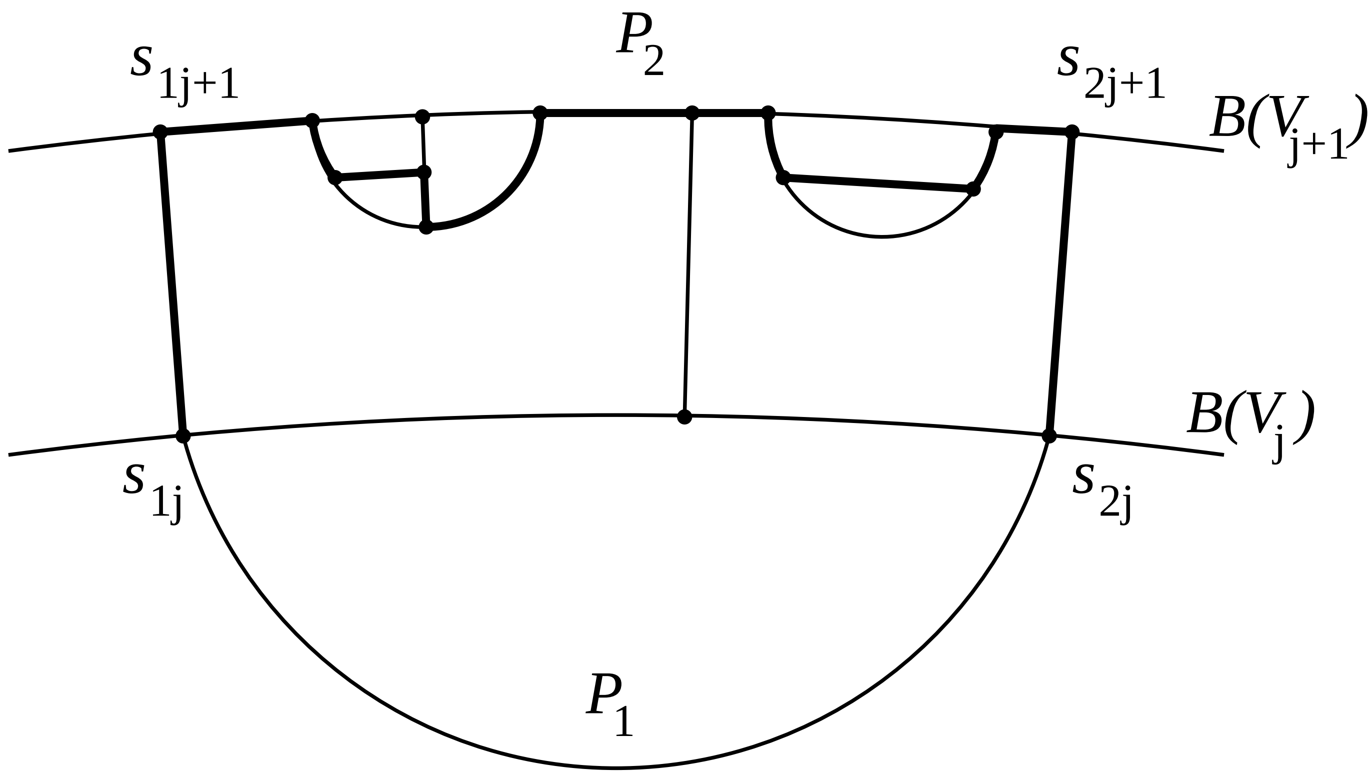

Fig. 3

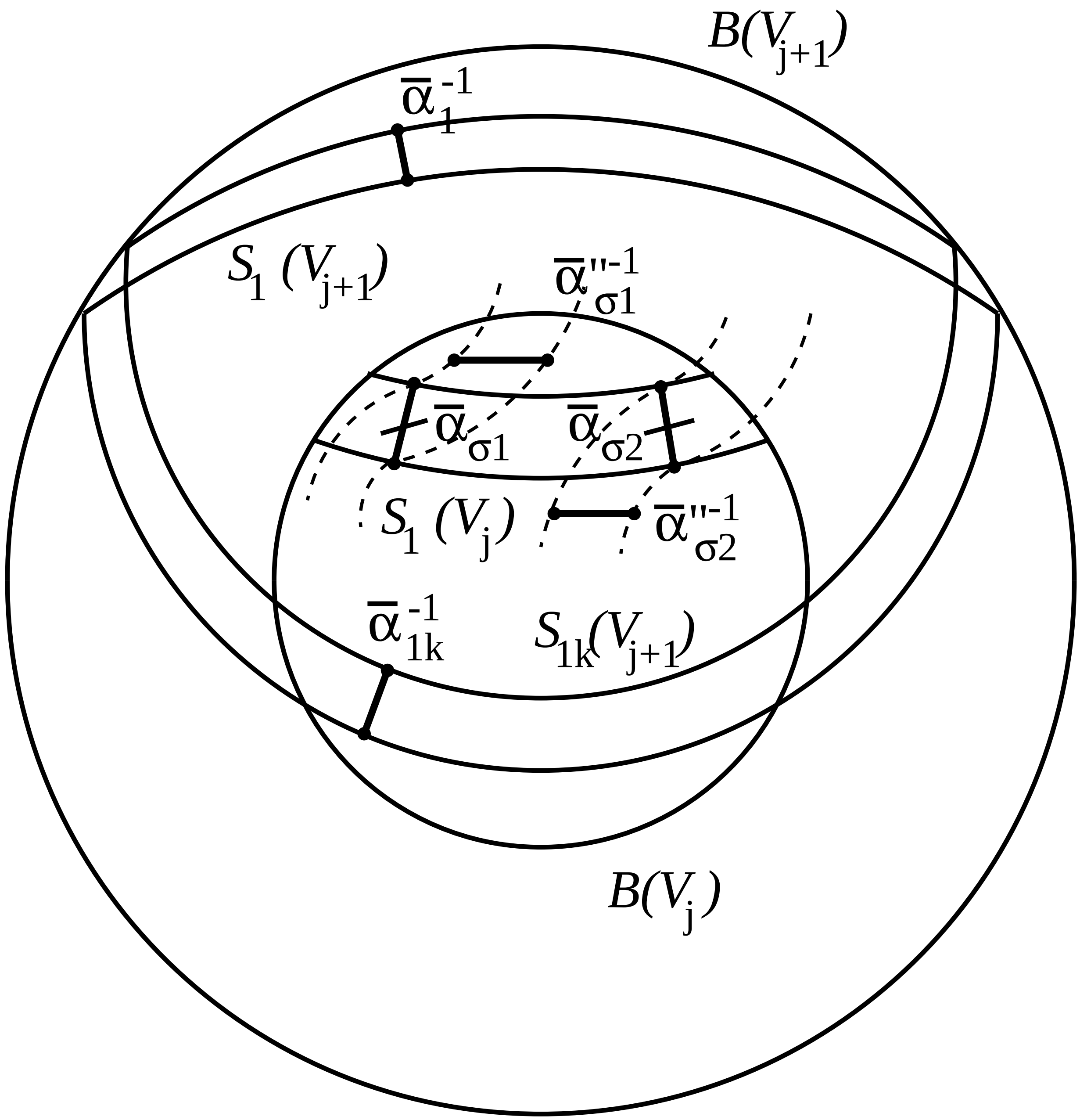

Fig. 3b

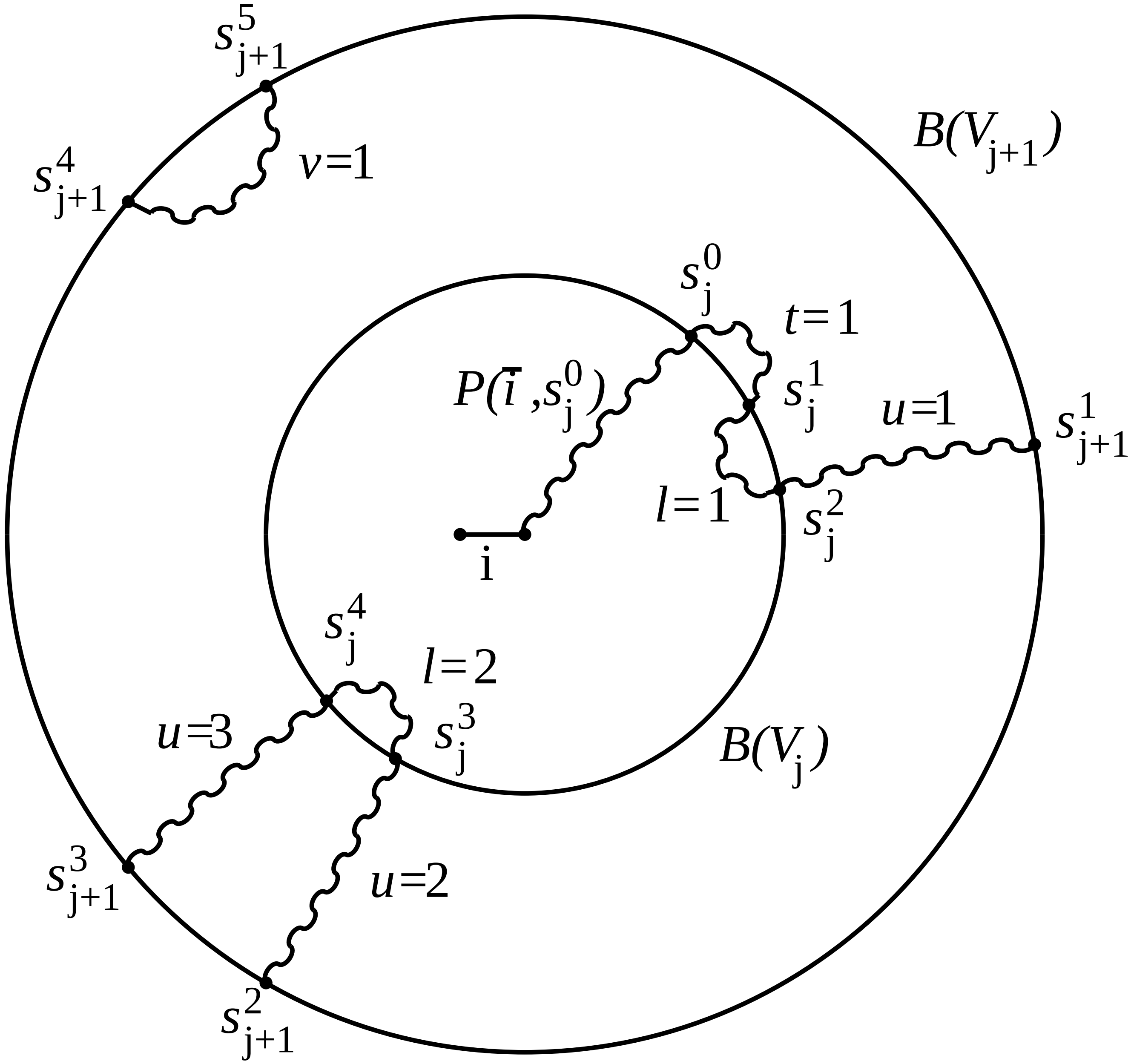
s_{j+1}^5
$B(V_{j+1})$
s_{j+1}^4
$v=1$
s_j^0
$t=1$
$P(\bar{i}, s_j^0)$
s_j^1
$u=1$
s_{j+1}^1
$l=1$
s_j^2
i
s_j^4
$l=2$
$u=3$
s_j^3
$B(V_j)$
s_{j+1}^3
$u=2$
s_{j+1}^2

Fig. 1